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PERSISTENCE OF HOMOCLINIC TANGENCIES IN HIGHER DIMENSIONS

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Abstract

In this paper we extend to a very general context Newhouse's phenomenon on the persistence of homoclinic tangencies and coexistence of infinitely many sinks. This is done using the corresponding results in codimension one recently proved by J. Palis and M. Viana, and in a reduction of codimension in the unfolding of homoclinic tangencies developed in the present paper.

1 Introduction.

Since Poincaré (1890) the concept of homoclinic orbit, an orbit at the intersection of stable and unstable manifolds from some periodic point, has became important for the development the theory of dynamical systems. Birkhoff in 1935 and Smale in 1965 described the dynamics to transversal homoclinic orbit. In fact, any transversal homoclinic orbit is always part of a hyperbolic Cantor set (horseshoes) whose orbit structure persists under small perturbations and its periodic orbits are dense.

About nontransversal homoclinic orbits or homoclinic tangencies when they are created and unfold, a great deal of changes happen in the dynamics

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of a diffeomorphism, creating a huge number of dynamical phenomena of "chaotic" type.

We say that a one-parameter family of diffeomorphisms $\{\varphi_\mu\}$, $\mu \in \mathbf{R}$, of a manifold M *unfolds generically a homoclinic tangency of codimension* $u \geq 1$ *in* $\mu = 0$ if φ_0 has a hyperbolic fixed (or periodic) point $p_0 \in M$ such that its stable and unstable manifolds, $W^s(p_0)$ and $W^u(p_0)$; with $dim W^u(p_0) = u$, has an orbit of tangency which, generically, we assume parabolic (quadratic contact) and for values of $\mu \in (0, \epsilon)$, transversal intersections are created between the "smooth" or "analytical" continuations, $W^s(p_\mu)$ and $W^u(p_\mu)$, of the stable and unstable manifolds of p_0, respectively. Moreover, these invariant manifolds move with nonvoid relative velocity together with the tangency when the parameter varies. In general, it can be assumed that the fixed point p_0 belong to a basic set Λ_0; and generically $|det(D\varphi_0(p_0))| \neq 1$. We said that the family $\{\varphi_\mu\}$, unfolds generically a homoclinic tangency of codimension one in $\mu = 0$ is *sectionally dissipative* if $|\sigma\lambda_i| < 1$ for each $1 \leq i \leq m-1$, $m = dimM$; where $\sigma, \lambda_1, \ldots, \lambda_{m-1}$ are the eigenvalues of $D\varphi_0(p_0)$, with $|\sigma| > 1 > |\lambda_i|$.

The problem is to understand the changes in the orbit structure of φ_μ, with $|\mu|$ small, with the creation of new transversal homoclinic orbits, as well as the type of dynamics which prevails in the unfolding. More precisely, to know the dynamics of the maximal invariant

$$\Sigma_\mu = \bigcap_{n \in \mathbf{Z}} \varphi_\mu^n(U \cup U_\mu),$$

where U is a neighbourhood of Λ_0 such that its hyperbolic analytical continuation Λ_μ satisfies

$$\Lambda_\mu = \bigcap_{n \in \mathbf{Z}} \varphi_\mu^n(U); \text{ and}$$

$$U_\mu = \bigcup_{|j| \leq N} \varphi_\mu^j(V_\mu)$$

for some $N > 0$. We take V_μ to be a neighbourhood from a point in the tangency orbit, whose points are at a distance of the order of K$|\mu|$ from the local stable manifold and from the unstable manifold of p_μ, the analytical continuation of p_0, where K is some positive constant.

The following results are known and related to this problem:

• **Cascades of period doubling bifurcations.** *If $\mu > 0$, or $\mu < 0$, is a parameter value for which φ_μ has Σ_μ as a hyperbolic set, given any periodic point $q_\mu \in \Sigma_\mu$ of period k, with index s = dim M - u and arbitrarily close to the tangency orbit, there exist $\mu_n \to 0$ such that φ_{μ_n} has in Σ_{μ_n} a periodic point of period $2^n k$ of index s + 1.*

(Yorke-Alligood, [YA], in codimension one; Martín Rivas, [M], in the general case).

• **Coexistency of infinitely many sinks.** *There are intervals I near $\mu = 0$ and residual sets $N \subset I$ such that φ_μ, $\mu \in N$, has infinitely many sinks. Moreover, if $\tau(\Lambda_0) = \tau^s(\Lambda_0)\tau^u(\Lambda_0) > 1$, the thickness of Λ_0, then some stable and unstable manifolds of points in Λ_μ, for every $\mu \in I$, with $\mu > 0$, or $\mu < 0$, are tangent to each other.*

(Newhouse and Robinson, [N] and [R], when dim M = 2; Palis-Viana, [PV], in general in the codimension one and sectionally dissipative case).

• **Abundance of strange attractors.** *Given $\epsilon > 0$ there are sets in the parameter space $E \subset (-\epsilon, \epsilon)$ with positive Lebesgue measure such that, for every $\mu \in E$, φ_μ has strange attractos of Hénon type in Σ_μ.*

(Mora-Viana, [MV], dim M = 2, based on Benedicks-Carleson, [BC]; Viana, [V], codimension one and sectionally dissipative).

• **Prevalence of hyperbolicity.** Denoting by $B(\varphi_\mu)$ the set of parameter values for which Σ_μ is not hyperbolic, $HD(\Lambda_0)$ the Hausdorff dimension of Λ_0 and m the Lebesgue measure, we have:

- *If $HD(\Lambda_0) < 1$, then*

$$\liminf_{\epsilon \to 0} \frac{m[B(\varphi_\mu) \cap (-\epsilon, \epsilon)]}{\epsilon} = 0$$

(Palis-Takens, [PT1], dim M = 2).

- *If $HD(\Lambda_0) > 1$, then*

$$\lim_{\epsilon \to 0} \frac{m[B(\varphi_\mu) \cap (-\epsilon, \epsilon)]}{\epsilon} > 0$$

(Palis-Yoccoz, [PY], dim M = 2).

Now we introduce a topological version of the results about coexistence of infinitely many sinks and persistence of tangencies. First, the following

Definition: We say that an open set $\mathcal{U} \subset Diff^2(M)$ has *persistent tangencies* if there are continuos maps Λ_1 and Λ_2 which assign to each φ in $\mathcal{U}$ basic sets $\Lambda_1(\varphi)$ and $\Lambda_2(\varphi)$ of saddle type, and for any $\varphi \in \mathcal{U}$ there is a tangency between $W^s(\Lambda_1(\varphi))$ and $W^u(\Lambda_2(\varphi))$ or such tangencies could be obtained by small perturbation of φ. When $\Lambda_1 = \Lambda_2$ we say that $\mathcal{U}$ is an open of *persistent homoclinic tangencies.*

Paraphrasing the result in [PV] in higher dimensions, we have

- *If $\varphi \in Diff^2(M)$ has a hyperbolic fixed point $p \in M$ whose stable and unstable manifolds have a codimension one orbit of tangency. Then, arbitrarily near φ there is an open set $\mathcal{U} \subset Diff^2(M)$ with persistent homoclinic tangencies. Furthermore, if the saddle point p is sectionally dissipative, there are residual sets $\mathcal{R} \subset \mathcal{U}$ such that each diffeomorphism in $\mathcal{R}$ display infinitely many sinks.*

The main purpose of this paper is to extend for the general context this last result mencioned. In fact, we prove

Theorem A. *Let $\varphi \in Diff^2(M)$ be a diffeomorphism on a manifolf M, dim $M \geq 3$, with a saddle point of index $1 \leq s \leq m-2$, whose stable and unstable manifolds have an orbit of tangency. Then, arbitrarily near φ there is an open set $\mathcal{U} \subset Diff^2(M)$ with persistent homoclinic tangencies. Moreover, if the eigenvalues of $D\varphi(p)$, $\lambda_1, \ldots, \lambda_s, \sigma_1, \ldots, \sigma_u$ are weakly dissipative, which means*

$$0 < |\lambda_s| \leq \ldots \leq |\lambda_1| < 1 < |\sigma_1| \leq \ldots \leq |\sigma_u|, \text{ and } |\sigma_1 \lambda_1| < 1, \qquad (1)$$

then there are residual sets $\mathcal{R} \subset \mathcal{U}$ with each diffeomorphism in $\mathcal{R}$ displaying infinitely many coexisting hyperbolic periodic orbits of index $s + 1$.

Theorem B. *Let $\{\varphi_\mu\}_\mu$ be a C^∞ one-parameter family of diffeomorphisms on M unfolding generically a quadratic homoclinic tangency of codimension $u \geq 2$ in $\mu = 0$. Let $p_0 \in M$ be the associated saddle point, of index s. There are intervals I_i of parameter values arbitrarily near $\mu = 0$, and residual subset $N_i \subset I_i$ such that, if the eigenvalues of $D\varphi_0(p_0)$ satisfies (1), for each $\mu \in N_i$, φ_μ has infinitely many hyperbolic periodic orbits of index s + 1.*

The proof of these theorems is based upon analogous results of Palis-Viana (cf. [PV]), and in a *reduction of codimension in homoclinic tangencies*, which it will be proved into next section. This result establishs:

Theorem C. *If $\{\varphi_\mu\}_\mu$ is a C^∞ generic family of diffeomorphisms on M unfolding generically a homoclinic tangency of codimension $u \geq 2$ in $\mu = 0$. Then,given $k \geq 3$ there is a sequence of intervals $I_n \to 0$ such that for every $\mu \in I_n$, φ_μ has a C^k submanifold $S_{\mu,n}$ of codimension u - 1 which is locally invariant, normally expansive and the family $\{\varphi_\mu\}_{\mu \in I_n}$ restricted to $S_{\mu,n}$ unfolds generically a homoclinic tangency of codimension one. Furthermore, if φ_0 satisfies (1), for every $\mu \in I_n$, φ_μ restricted to $S_{\mu,n}$ is sectionally dissipative.*

There is a technical point to be observed in the direct use of Palis-Viana results in the previous theorem. Since center-stable theorems are not C^∞ in general, we have to remove some linearizability condition by they use, although they are aware that it is not needed. Anyhow, we remove this assumption altogether in section 3.

As a consequence of the reduction of codimension in homoclinic tangencies, the following are obtained as well:

• *Existence of cascades of period doubling bifurcations in any codimension.*

• *Abundance of what we call strange saddle-sets in generic families unfolding a homoclinic tangency of codimension $u \geq 2$.*

These strange saddle-sets arise naturally as normally expanding Hénon-like strange attractors obtained in codimension-one invariant submanifolds provided by Theorem C.

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2 Reduction of Codimension in Homoclinic Tangencies.

Let us consider a C^∞ one-parameter family of diffeomorphisms $\varphi_\mu : M \to M$, where M is a C^∞ manifold of dimension $m \geq 3$, such that in $\mu = 0$ a homoclinic parabolic tangency of codimension $u \geq 2$ is generically unfolds, which is associated with a hyperbolic fixed (or periodic) point $p_0 \in M$ of index s$\geq$ 1. This means, see [NPT], that near a point r in the orbit of tangency there are μ-dependents coordinates $(x_1, \dots, x_m)$, in which

$$\mathrm{r} = (0, \dots, 0);$$

$$W^s(p_\mu) = \{x_{s+1} = \dots = x_m = 0\}; \text{ and}$$

$$W^u(p_\mu) = \{x_1 = \dots = x_{m-(u+1)} = 0, x_m = -x_{m-u}^2 + \mu\};$$

where p_μ is the analytical continuation of p_0.

Among the generic conditions to be considered for such families, we assume that the eigenvalues of $D\varphi_0(p_0)$, $\lambda_1, \dots, \lambda_s, \sigma_1, \dots, \sigma_u$ are simple and satisfy:

i) they are weakly dissipative, see (1);

ii) if $\sigma_1 \in \mathbf{R}$, we assume positive and $|\sigma_2| > \sigma_1$; if $\sigma_1 \notin \mathbf{R}$, then $\sigma_2 = \bar{\sigma_1}$ and $|\sigma_1| < |\sigma_3|$; and

iii) for $k \geq 3$ fixed, there are C^k-linear coordinates, μ-dependents, defined in a neighbourhood V of p_0, let us say $\xi = (\xi_1, \dots, \xi_s)$ and $\eta = (\eta_1, \dots, \eta_u)$, see [S] and [T].

Under such conditions, for $(\xi, \eta) \in \varphi_\mu^{-1}(V) \cap V$ we have

$$\varphi_\mu(\xi, \eta) = (E_\mu \xi, U_\mu \eta);$$

where E_μ (resp. U_μ) is a square matrix of the form

$$E_\mu = \mathrm{diag}(\widetilde{\lambda}_1, \dots, \widetilde{\lambda}_s)\widetilde{E}_\mu, \text{ (resp. } U_\mu = \mathrm{diag}(\widetilde{\sigma}_1, \dots, \widetilde{\sigma}_u)\widetilde{U}_\mu),$$

with $\widetilde{E}_\mu$ and $\widetilde{U}_\mu$ are orthogonal, and

$$\widetilde{\lambda}_j = \begin{cases} \lambda_j & \text{if } \lambda_j \in \mathbf{R} \\ |\lambda_j| & \text{if } \lambda_j \notin \mathbf{R} \end{cases},$$

in the same way we define $\widetilde{\sigma}_j$.

Let $r = (\xi_0, 0^u) \in W^s_{loc}(p_0) \cap V$ and $q = (0^s, \eta_0) \in W^u_{loc}(p_0) \cap V$ be points in the orbit of tangency; and let $N > 0$ be such that $\varphi_0^N(q) = r$. Then for μ, ξ and η small enough, $\varphi_\mu^N(\xi, \eta + \eta_0) \in V$; afterwards in the (ξ, η) coordinates it is written

$$\varphi_\mu^N(\xi, \eta + \eta_0) = (H(\mu, \xi, \eta), G(\mu, \xi, \eta));$$

obviously $H(0, 0^s, 0^u) = \xi_0$ and $G(0, 0^s, 0^u) = 0^u$.

Let us characterize the quadratic contact of the tangency and genericity of its unfold in terms of H ang G. Except otherwise the opposite, the partial derivatives are calculeted at $\mu = 0$, $\xi = 0^s \in \mathbf{R}^s$ and $\eta = 0^u \in \mathbf{R}^u$

From the quasi-transversal intersection in the orbit of tangency, there is only one direction $W \subset T_q W^u(p_0)$ where $D\varphi_0^N(q)(W)$ is the common direction between $T_r W^s(p_0)$ and $T_r W^u(p_0)$; we recall that $Z_r = T_r W^s(p_0) + T_r W^u(p_0)$ is of codimension one, see [NPT].

Let $(0^s, v = (v_1 \ldots, v_u))$ be a non-null vector in W; denoting $D\varphi_0^N$ by the matrix

$$\begin{bmatrix} \partial_\xi H & \partial_\eta H \\ \partial_\xi G & \partial_\eta G \end{bmatrix}$$

then, we have

$$\sum_{i=1}^{u} \partial_{\eta_i} G_k v_i = 0 \ , k = 1, \ldots, u; \tag{2}$$

where G_k is the k^{th}-component of G.

Assuming $v_1 \neq 0$, $\partial = (0^s, (a_1, \ldots, a_u))$ with $a_i = (-1)^{1+i} det \partial_\eta G(i/1)$, is orthogonal to Z_r; and $\{(e_1, 0^u), \ldots, (e_s, 0^u), \partial_{\eta_2}\varphi_0^N(q), \ldots, \partial_{\eta_u}\varphi_0^N(q), \partial\}$ is a base for $T_r M$; e_i denote the i^{th}-canonical vector of $\mathbf{R}^s$ and $\partial_\eta G(i/1)$ is the matrix obtained from $\partial_\eta G$ eliminating row i and column 1. There are also C^k curves $C_\mu \subset W^u_{loc}(p_\mu) \cap V$ with parametric equations

$$C_\mu : q + (0^s, (h^1_\mu(t), \ldots, h^u_\mu(t)), |t| \leq \epsilon_0 ;$$

for some $\epsilon_0 > 0$, with

$$h_0^i(0) = 0 \text{ and } \tfrac{d}{dt}h_0^i = v_i,\ \text{i=1, \dots, u} \ ;$$

and unless otherwise a readaptation in the coordinates and a reparametrization, the ∂-coordinate of $\varphi_\mu^N(q)$ is μ. Furthermore, this value is a relative extreme, we assume maximum, in the ∂-coordinate restricted to $\varphi_\mu^N(C\mu)$. In fact, the ∂-coordinate of $q + (0^s, (h_\mu^1(t), \dots, h_\mu^u(t)))$ is

$$\alpha(\mu, t) = \frac{1}{\|\partial\|^2} \sum_{i=1}^{u} a_i G_i(\mu, 0^s, (h_\mu^1(t), \dots, h_\mu^u(t))),$$

to simplify, ∂ it is supposed unitary.

In this way, from the genericity of the tangency and its unfolding, follow:

$$\sum_{i=1}^{u} a_i G_i(\mu, 0^s, 0^u) = \mu; \tag{3}$$

$$\sum_{i=1}^{u} a_i \partial_{\mu\mu} G_i = 0; \tag{4}$$

$$\sum_{i=1}^{u} a_i \sum_{j=1}^{u} \partial_{\mu\eta_j} G_i v_j = 0; \text{ and} \tag{5}$$

$$\sum_{i=1}^{u} a_i L_i(v, v) < 0; \tag{6}$$

where L_i is the quadratic form $\partial_{\eta\eta} G_i$, $i = 1, \dots, u$.

Proof of Theorem C: For each $n \geq 1$ and μ near zero, let $(b_1, \dots, b_u)$ be the vector given by

$$(b_1, \dots, b_u)\widetilde{U}_\mu^n = (a_1, \dots, a_u) \tag{7}$$

The generic condition is impossed :

$$b_1 \neq 0; \sum_{i=1}^{k_1} b_i v_i \neq 0 \text{ and } \sum_{i=1}^{k_1} b_i \eta_i > 0; \tag{8}$$

where k_1 is the dimension of the weak expansion of $W^u(p_0)$, and $(\eta_1, \ldots, \eta_u) = \eta_0$.

Denoting $V_0 = \{(\xi, \eta) : \|\xi - \xi_0\| \leq 1 \, , \, \|\eta - \eta_0\| \leq 1\}$, we assume without generality lost $V_0 \subset V$. In V_0 are defined diffeomorphisms over its respective images, $V^u_{\mu,n}$ and $V^s_{\mu,n}$, given by

$$\Psi^u_{\mu,n}(\xi, \eta) = (E^n_\mu \xi, \eta) \text{ and } \Psi^s_{\mu,n}(\xi, \eta) = (\xi, U^{-n}_\mu \eta).$$

It is observed that $V^s_{\mu,n}$ is the set of points $(\xi, \eta) \in \mathbf{R}^s \times \mathbf{R}^u$ such that $(\xi, U^n_\mu \eta) \in V_0$, $\varphi^n_\mu(V^s_{\mu,n}) = V^u_{\mu,n}$ and $\varphi^n_\mu(\xi, \eta) = \Psi^u_{\mu,n} \circ (\Psi^s_{\mu,n})^{-1}(\xi, \eta)$.

Now we define an n-dependent reparametrization and μ-dependent coordinate transformation on V_0 by the following formulas:

$$\Psi_n(a; x, y) = (\mu = \mu(a, n), \Psi_{a,n}(x, y)),$$

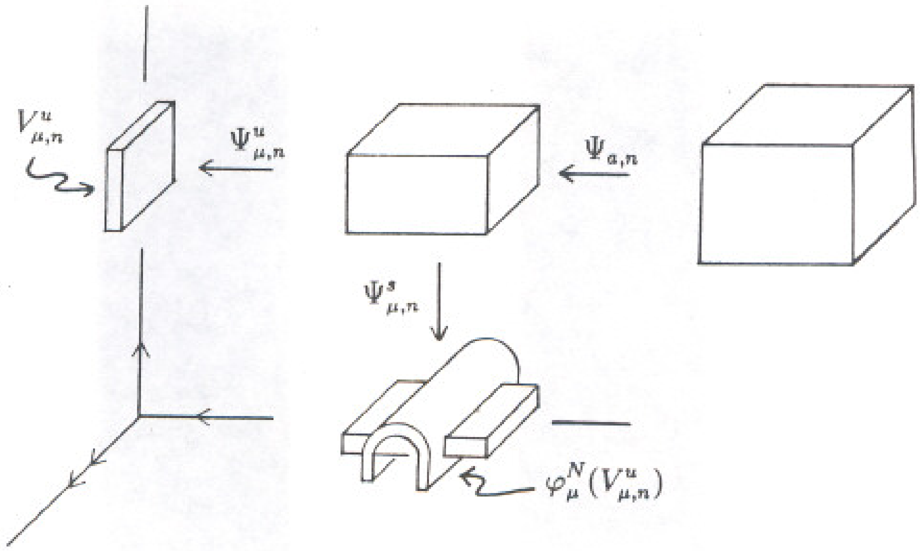

Figure 2.1

where

$$\mu = \frac{1}{1 + \tilde{\sigma}_1^{-n} L^1_{a,n}(M_n^{-1} C_n)} \left[- \frac{\left(\sum_{i=1}^{k_1} b_i v_i (1 + \theta_n)\right)^2}{\sum_{i=1}^{u} a_i L_i(v, v)} \tilde{\sigma}_1^{-2n} a \right.$$
$$\left. - \tilde{\sigma}_1^{-2n} Q_{a,n}(M_n^{-1} C_n) - \sum_{i=1}^{u} a_i \partial_\xi G_i E^n_\mu \xi_0 + \tilde{\sigma}_1^{-n} \sum_{i=1}^{k_1} b_i \eta_i \right.$$

$$-\tilde{\sigma}_1^{-2n}\left(\frac{\sum_{i=1}^{k_1} b_i v_i(1+\theta_n)}{\sum_{i=1}^{u} a_i L_i(v,v)}\right)\left(\frac{L_{a,n}^2(M_n^{-1}C_n)}{2}+\frac{\left(L_{a,n}^2(M_n^{-1}C_n)\right)^2}{4}\right)\Bigg]$$

$$\Psi_{a,n}(x,y) = (\beta_n x + \xi_0, B_n y + \eta_0);$$

where $\beta_n = \tilde{\sigma}_1^{-n}\omega^n$ with $1 < \omega < \min\{\tilde{\sigma}_1, |\lambda_1\sigma_1|^{-1}\}$;

$$B_n = \begin{bmatrix} dv_1\tilde{\sigma}_1^{-n} & -\rho_1^{-n}b_2/b_1 & \cdots & -\rho_1^{-n}b_u/b_1 \\ dv_2\tilde{\sigma}_1^{-n} & \rho_2^{-n} & & \bigcirc \\ \vdots & & \ddots & \\ dv_u\tilde{\sigma}_1^{-n} & \bigcirc & & \rho_u^{-n} \end{bmatrix}$$

being that

$$d = -\frac{a\sum_{i=1}^{k_1} b_i v_i(1+\theta_n)}{\sum_{i=1}^{u} a_i L_i(v,v)} \text{ and } \rho_j = \tilde{\sigma}_1\tilde{\sigma}_u\tilde{\sigma}_j^{-1},\ j = 1,\ldots,u;$$

$M_{a,n}$ is a non-singular matrix and for $a \in J$, compact interval,

$$0 < \inf_{a,n} \|M_{a,n}\| \leq \sup_{a,n} \|M_{a,n}\| < \infty \text{ (see Lemma 2.1);}$$

$C_{a,n} \in \mathbf{R}^{u-1}$ and for $a \in J$ we have

$$\sup_{a,n} \|C_{a,n}\| < \infty \text{ (see Lemma 2.1);}$$

$L_{a,n}^1$ and $L_{a,n}^2$ are linear form with

$$\sup_{a,n} \|L_{a,n}^i\| < \infty, i = 1,2 \text{ (see Lemma 2.1);}$$

$Q_{a,n}$ is a quadratic form and

$$\sup_{a,n} \|Q_{a,n}\| < \infty \text{ (see Lemma 2.1)}$$

It must be observed that for $a \in J$, and n sufficiently large, $\mu = O\left(\tilde{\sigma}_1^{-n}\right)$. We define now

$$\varphi_n(a;x,y) = \left(\mu; \varphi_{a,n} = \Psi_{\mu,n}^u \circ \Psi_{a,n}(x,y)\right) \textit{ and}$$

$$\Phi_{a,n}(x,y) = \varphi_{a,n}^{-1} \circ \varphi_\mu^{n+N} \circ \varphi_{a,n}(x,y) = (\bar{H}_{a,n}(x,y), \bar{G}_{a,n}(x,y)),$$

where

$$\bar{H}_{a,n}(x,y) = \beta_n^{-1}\left(H(\mu, E_\mu^n(\beta_n x + \xi_0), B_n y) - \xi_0\right), \text{ and}$$

$$\bar{G}_{a,n}(x,y) = B_n^{-1}\left(U_\mu^n G(\mu, E_\mu^n(\beta_n x + \xi_0), B_n y) - \eta_0\right).$$

We denote every $y \in \mathbf{R}^u$ by $y = (y_1, \bar{y}) \in \mathbf{R} \times \mathbf{R}^{u-1}$; and $\bar{G}_{a,n} = (G_{a,n}^1, G_{a,n}^2)$, being that $G_{a,n}^1 \in \mathbf{R}$ and $G_{a,n}^2 \in \mathbf{R}^{u-1}$.

Lemma 2.1. *a) For every $a \in J$, compact interval, $\bar{H}_{a,n}$ converges to zero when $n \to \infty$ in the C^k topology (uniformly on compacta)*

b) The maps $G_{a,n}^1$ and $G_{a,n}^2$ have the form:

$$G_{a,n}^1(x, y_1, \bar{y}) = k_{a,n} + \ell_{a,n}^1(\bar{y}) + q_{a,n}(\bar{y}) + \ell_{a,n}^2(\bar{y})y_1 - a y_1^2 + R_{a,n}^1(x, y_1, \bar{y});$$

$$G_{a,n}^2(x, y_1, \bar{y}) = \tilde{\sigma}_u^n(C_{a,n} + M_{a,n}\bar{y}) + R_{a,n}^2(x, y_1, \bar{y});$$

where

$\ell_{a,n}^1$ and $\ell_{a,n}^2$ are linear forms such that

$$\sup_{a,n} \|\ell_{a,n}^i\| < \infty \ , \ i = 1, 2;$$

$q_{a,n}$ is a quadratic form and

$$\sup_{a,n} \|q_{a,n}\| < \infty;$$

$$k_{a,n} = 1 + \ell_{a,n}^1(M_n^{-1}C_n) - q_{a,n}(M_n^{-1}C_n) - \frac{\ell_{a,n}^2(M_n^{-1}C_n)}{2a} - \frac{\left(\ell_{a,n}^2(M_n^{-1}C_n)\right)^2}{4a};$$

$R_{a,n}^1, \tilde{\sigma}_u^{-n} R_{a,n}^2$ converge to zero when $n \to \infty$ in the C^k topology (uniformly compacta); and $M_{a,n}$ and $C_{a,n}$ as before .

Proof. a) The conclusion follows inmediately after the following facts:

$$\begin{aligned} \bar{H}_{a,n}(x,y) &= \mu\beta_n^{-1}\partial_\mu H + \beta_n^{-1}\partial_\xi H E_\mu^n \xi_0 + \beta_n^{-1}\partial_\eta H B_n y \\ &+ \partial_\xi H E_\mu^n x + \beta_n^{-1} r_n(\mu, E_\mu^n(\beta_n x + \xi_0), B_n y); \end{aligned}$$

where $r_n, \partial_\mu r_n, \partial_\xi r_n$, and $\partial_\eta r_n$ are null in $\mu = 0, \xi = 0^s$ and $\eta = 0^u$; $\mu = O(\tilde{\sigma}_1^{-n}), |\lambda_1 \sigma_1| < 1$ and $|\beta_n^{-1} \sigma_1^{-n}| \to 0$ for $n \to \infty$.

b) It is clear that

$$\begin{aligned} G^1_{a,n}(x,y) &= \sum_{i=1}^{u} \langle F_1(B_n^{-1}), C_i(U_\mu^n)\rangle G_i(\mu, E_\mu^n(\beta_n x + \xi_0), B_n y) \\ &\quad -\langle F_1(B_n^{-1}), \eta_0\rangle; \end{aligned}$$

here, $F_1(B)$ and $C_i(B)$ denote the row one and i-column of matrix B, and $\langle , \rangle$ is the Euclidean product.

One can easily verify that

$$F_1(B_n^{-1}) = \frac{\tilde{\sigma}_1^{2n}}{d \sum_{i=1}^{k_1} b_i v_i (1+\theta_n)} (b_1 \tilde{\sigma}_1^{-n}, \ldots, b_u \tilde{\sigma}_u^{-n});$$

with $\theta_n \to 0$ when $n \to \infty$. Then,

$$\langle F_1(B_n^{-1}), C_\ell(U_\mu^n)\rangle = \frac{\tilde{\sigma}_1^{2n}}{d \sum_{i=1}^{k_1} b_i v_i (1+\theta_n)} a_\ell \ for\ \ell = 1, \ldots, u; \tag{9}$$

$$\langle F_1(B_n^{-1}), \eta_0\rangle = \frac{\tilde{\sigma}_1^{n} \sum_{i=1}^{k_1} b_i \eta_i}{d \sum_{i=1}^{k_1} b_i v_i (1+\theta_n)} + \bar{\theta}_n; \tag{10}$$

where $\bar{\theta}_n \to 0$ when $n \to \infty$.

On the other hand,

$$\begin{aligned} G_\ell(\mu, E_\mu^n(\beta_n x + \xi_0), B_n y) &= \mu \partial_\mu G_\ell + \partial_\xi G_\ell E_\mu^n(\beta_n x + \xi_0) + \partial_\eta G_\ell B_n y \\ &+ \mu \partial_{\mu\xi} G_\ell E_\mu^n(\beta_n x + \xi_0) + \frac{1}{2} \partial_{\xi\xi} G_\ell (E_\mu^n(\beta_n x + \xi_0), E_\mu^n(\beta_n x + \xi_0)) \\ &+ \frac{1}{2}\mu^2 \partial_{\mu\mu} G_\ell + \mu \partial_{\mu\eta} G_\ell B_n y + \partial_{\eta\xi} G_\ell (B_n y, E_\mu^n(\beta_n x + \xi_0)) \\ &+ \frac{1}{2} L_\ell(B_n y, B_n y) + \bar{G}_\ell(\mu, E_\mu^n(\beta_n x + \xi_0), B_n y); \end{aligned}$$

where $\bar{G}_\ell$ is C^k and vanishes together with its partial derivatives up to order two in $\mu = 0, \xi = 0^s$ and $\eta = 0^u$. Besides, from $|\beta_n \tilde{\sigma}_1^{2n}| \, \|E_\mu^n\| \to 0$ when $n \to$

∞, $|\lambda_1\sigma_1| < 1$ and $\mu = O(\tilde{\sigma}_1^{-n})$, we have for any multi-index α in (x, y), with $0 \le |\alpha| \le k$, that

$$|\sigma_1|^{2n}\partial^\alpha \Big[\beta_n \partial_\xi G_\ell E^n_\mu(\beta_n x + \xi_0) + \mu \partial_{\mu\xi} G_\ell E^n_\mu(\beta_n x + \xi_0)$$
$$+\partial_{\eta\xi} G_\ell(B_n y, E^n_\mu(\beta_n x + \xi_0)) + \frac{1}{2}\partial_{\xi\xi} G_\ell(E^n_\mu(\beta_n x + \xi_0), (E^n_\mu(\beta_n x + \xi_0))$$
$$+\bar{G}_\ell(\mu, E^n_\mu(\beta_n x + \xi_0), B_n y)\Big] \to 0$$

when $n \to \infty$ on compact parts.

Observing that

- $$\partial_\eta G_\ell B_n y = d\tilde{\sigma}_1^{-n} y_1 \sum_{i=1}^{u} \partial_{\eta_i} G_\ell v_i - \frac{\rho_1^{-n}}{b_1}\partial_{\eta_1} G_\ell \sum_{i=2}^{u} b_i y_i + \sum_{i=2}^{u} \rho_i^{-n} \partial_{\eta_i} G_\ell y_i;$$

- $$\partial_{\mu\eta} G_\ell B_n y = d\tilde{\sigma}_1^{-n} y_1 \sum_{i=1}^{u} \partial_{\mu\eta_i} G_\ell v_i + \tilde{\sigma}_1^{-n} L^1_{a,n,\ell}(\bar{y});$$ and

- $$L_\ell(B_n y, B_n y) = d^2\tilde{\sigma}_1^{-2n} y_1^2 L_\ell(v, v) + d\tilde{\sigma}_1^{-2n} y_1 L^2_{a,n,\ell}(\bar{y}) + \tilde{\sigma}_1^{-2n} Q_{a,n,\ell}(\bar{y});$$

where $L^i_{a,n,\ell}$, $i = 1, 2$ and $Q_{a,n,\ell}$ are respectively linear and quadratic forms such that

$$\sup_{a,n} \|L_{a,n,\ell}\| < \infty,\ i = 1, 2\ ,\ \text{and}\ \sup_{a,n} \|Q_{a,n,\ell}\| < \infty.$$

We have from the orthogonality of ∂ and Z_r, and (3), ..., (10) that

$$G^1_{a,n}(x, y_1, \bar{y}) = \frac{\tilde{\sigma}_1^{2n}}{d\sum_{i=1}^{k_1} b_i v_i (1 + \theta_n)}[\mu + \sum_{i=1}^{u} a_i \partial_\xi G_i E^n_\mu \xi_0.$$
$$+\tilde{\sigma}_1^{-n}\mu L^1_{a,n}(\bar{y}) + d^2\tilde{\sigma}_1^{-2n} y_1^2 \sum_{i=1}^{u} a_i L_i(v, v) + d\tilde{\sigma}_1^{-2n} L^2_{a,n}(\bar{y}) y_1$$
$$+\tilde{\sigma}_1^{-2n} Q_{a,n}(\bar{y})] - \frac{\tilde{\sigma}_1^n \sum_{i=1}^{k_1} b_i \eta_i}{d\sum_{i=1}^{k_1} b_i v_i (1 + \theta_n)} + r^1_{a,n}(x, y_1, \bar{y});$$

where

$$L^j_{a,n}(\bar{y}) = \sum_{i=1}^{u} L^j_{a,n,i}(\bar{y}), \text{ j=1,2; and } Q_{a,n}(\bar{y}) = \sum_{i=1}^{u} a_i Q_{a,n,i}(\bar{y}),$$

(used in the definition of μ), and $r^1_{a,n}$ converging to zero when $n \to \infty$ in the C^k topology (uniformly compacta).

From the definition of the reparametrization it is easy to verify

$$\frac{\tilde{\sigma}_1^{2n}}{d\sum_{i=1}^{k_1} b_i v_i(1+\theta_n)}\left[\mu + \sum_{i=1}^{u} a_i \partial_\xi G_i E^n_\mu \xi_0 - \tilde{\sigma}_1^{-n}\sum_{i=1}^{k_1} b_i \eta_i\right]$$

$$= 1 + \frac{1}{d\sum_{i=1}^{k_1} b_i v_i(1+\theta_n)}\left[\tilde{\sigma}_1^n \mu L^1_{a,n}(M^{-1}_{a,n}C_{a,n}) - Q_{a,n}(M^{-1}_{a,n}C_{a,n})\right.$$

$$\left. - \frac{d}{2a} L^2_{a,n}(M^{-1}_{a,n}C_{a,n}) - \frac{d}{4a}\left(L^2_{a,n}(M^{-1}_{a,n}C_{a,n})\right)^2\right].$$

Then,

$$G^1_{a,n}(a, y_1, \bar{y}) = 1 - \frac{L^2_{a,n}(M^{-1}_{a,n}C_{a,n})}{2a\sum_{i=1}^{k_1} b_i v_i(1+\theta_n)} - \frac{1}{4a}\left(\frac{L^2_{a,n}(M^{-1}_{a,n}C_{a,n})}{\sum_{i=1}^{k_1} b_i v_i(1+\theta_n)}\right)^2$$

$$+\frac{1}{d\sum_{i=1}^{k_1} b_i v_i(1+\theta_n)}\left[\tilde{\sigma}_1^n \mu L^1_{a,n}(\bar{y} + M^{-1}_{a,n}C_{a,n}) + Q_{a,n}(\bar{y}) - Q_{a,n}(M^{-1}_{a,n}C_{a,n})\right]$$

$$+\frac{L^2_{a,n}(\bar{y})}{\sum_{i=1}^{k_1} b_i v_i(1+\theta_n)} y_1 - a y_1^2 + R^1_{a,n}(x, y_1, \bar{y});$$

in which $R^1_{a,n} \to 0$ in the C^k topology on compact parts.
Making

$$\ell^1_{a,n} = \frac{\sigma_1^n \mu L^1_{a,n}}{d\sum_{i=1}^{k_1} b_i v_i(1+\theta_n)}, \ \ell^2_{a,n} = \frac{L^2_{a,n}}{\sum_{i=1}^{k_1} b_i v_i(1+\theta_n)} \text{ and}$$

$$q_{a,n} = \frac{Q_{a,n}}{d\sum_{i=1}^{k_1} b_i v_i(1+\theta_n)},$$

we have the first part of b).

It is easy to verify that

$$G^2_{a,n}(x,y) = \mu \tilde{B}_n U^n_\mu \partial_\mu G + \tilde{B}_n U^n_\mu \partial_\xi G E^n_\mu(\beta_n x + \xi_0)$$

$$+\tilde{B}_n U_\mu^n \partial_\eta G B_n y + \tilde{B}_n U_\mu^n \tilde{G}(\mu, E_\mu^n(\beta_n x + \xi_0), B_n y) - \tilde{B}_n \eta_0;$$

where $\tilde{G}$ is C^k and vanishes together its partial derivatives in $\mu = 0$, $\xi = 0^s$ and $\eta = 0^u$.

We make the following observations:

• $\partial_\eta G B_n y = \partial_\eta \bar{B}_n \bar{y}$ for every $y \in \mathbf{R}^u$,this is a consequence of $\partial_\eta G v = 0$; with $\bar{B}_n$ being the matrix B_n without column one.

• Using (8), the orthogonality of $(a_1, \ldots, a_u)$ with the image of $\partial_\eta G$ and $\rho_i \tilde{\sigma}_i = \tilde{\sigma}_1 \tilde{\sigma}_u$ for $i = 1, \ldots, u$; follow

$$\tilde{B}_n U_\mu^n \partial_\eta G = (\tilde{\sigma}_1 \tilde{\sigma}_u)^n \, E \tilde{U}_\mu^n \partial_\eta G,$$

where

$$E = \begin{bmatrix} 0 & 1 & & \bigcirc \\ & & \ddots & \\ \bigcirc & & & 1 \end{bmatrix}_{(u-1)xu}$$

• As $\tilde{U}_\mu$ is orthogonal, the image of $\tilde{U}_\mu^n \partial_\eta G \bar{B}_n$is orthogonal to $(b_1, \ldots, b_u)$, in particular this imply that the matrix $M_{a,n}$ bellow is non-singular.

A simple calculation shows

$$\|\bar{B}_n\| = \delta_n \tilde{\sigma}_1^{-n} \text{ with } 0 < \inf_n \delta_n \leq \sup_n \delta_n < \infty;$$

$$\|\tilde{B}_n\| \leq O(|\rho_2|^n) \text{ and } \|\tilde{B}_n U_\mu^n\| \leq O(|\sigma_1 \sigma_u|^n),$$

then $M_{a,n} = \tilde{\sigma}_1^n E \tilde{U}_\mu^n \partial_\eta G \bar{B}_n$ and $C_{a,n} = \mu \tilde{\sigma}_1^{-n} \tilde{B}_n U_\mu^n \partial_\mu G - \tilde{B}_n \eta_0$ are bounded and $M_{a,n}$ is bounded away zero.

In this manner

$$G_{a,n}^2(x, y_1, \bar{y}) = \tilde{\sigma}_u^n (C_{a,n} + M_{a,n} \bar{y}) + R_{a,n}^2(x, y_1, \bar{y});$$

where

$$R_{a,n}^2(x, y_1, \bar{y}) = \tilde{B}_n U_\mu^n \left(\partial_\xi G E_\mu^n(\beta_n x + \xi_0) + \tilde{G}(\mu, E_\mu^n(\beta_n x + \xi_0), B_n y \right);$$

obviously $\tilde{\sigma}_u^{-n} R_{a,n}^2$ going to zero in the C^k topology (uniformly compacta), when $n \to \infty$. □

2.1 Existence of locally invariant graphs

It is clear that any map $h : domh \subset \mathbf{R}^s \times \mathbf{R} \to \mathbf{R}^{u-1}$ whose graph is $\Phi_{a,n}$-invariant satisfies, according to the previous lemma, the functional equation

$$h(x,y_1) = \tilde{\sigma}_1^{-n} M_{a,n}^{-1}[h(\bar{H}_{a,n}(x,y_1,h(x,y_1)), G_{a,n}^1(x,y_1,h(x,y_1))) \qquad (11)$$
$$-R_{a,n}^2(x,y_1,h(x,y_1))] - M_{a,n}^{-1}C_{a,n}.$$

Though, since graphs of local maps are not necessarily transformed by $\Phi_{a,n}$ in graphs of maps having at least the same domain, we will proceed to study an extended version of (11). For $a \in J$, compact interval, and n sufficiently large, we consider extensions $\tilde{H}_{a,n}$, $\tilde{G}_{a,n}^1$ and $\tilde{R}_{a,n}^2$ of $\bar{H}_{a,n}$, $G_{a,n}^1$ and $R_{a,n}^2$ respectively, coinciding in $Q = [-T,T]^s x[-T,T]x[-T,T]^{u-1}$ for some $T > 0$, and for some constant $K > 0$

$$\|\tilde{H}_{a,n}\|_k \leq K\|\bar{H}_{a,n} \mid Q\|_k \;;\; \|\tilde{G}_{a,n}^1\|_k \leq K\|G_{a,n}^1 \mid Q\|_k \text{ and } \|\tilde{R}_{a,n}^2\|_k \leq K\|R_{a,n}^2\|_k.$$

We study the problem of finding globally defined solutions of

$$h(x,y_1) = \tilde{\sigma}_1^{-n} M_{a,n}^{-1}[h(\tilde{H}_{a,n}(x,y_1,h(x,y_1)), \tilde{G}_{a,n}^1(x,y_1,h(x,y_1))) \qquad (12)$$
$$-\tilde{R}_{a,n}^2(x,y_1,h(x,y_1))] - M_{a,n}^{-1}C_{a,n}.$$

It is easy to verify that bounded solutions of (12) satisfy $\|h + M_{a,n}^{-1}C_{a,n}\| \leq \gamma_{a,n}$, where $\gamma_{a,n} \to 0$ when $n \to \infty$; in fact,

$$\gamma_{a,n} = O(max\{|\sigma_u|^{-n}, \|\tilde{\sigma}_u^{-n}\tilde{R}_{a,n}^2\|\}).$$

Let

$$\Theta_{a,n} = \{h \in C(\mathbf{R}^s \times \mathbf{R}, \mathbf{R}^{u-1}) : \|h + M_{a,n}^{-1}C_{a,n}\| \leq \gamma_{a,n} \text{ and } L(h) \leq 1\},$$

where $L(h)$ is the Lipschitz's constant of h. Cleary $\Theta_{a,n}$ is complete with the supreme norm; moreover, $\Gamma_{a,n} : \Theta_{a,n} \to \Theta_{a,n}$ given by (12) is a well defined operator and for n big enough is a contraction. If $h_{a,n}$ is the fixed point of $\Gamma_{a,n}$, then $h_{a,n}$ defines an invariant graphic for

$$\tilde{\Phi}_{a,n} = (\tilde{H}_{a,n}, \tilde{G}_{a,n}^1, \tilde{G}_{a,n}^2);$$

where

$$\tilde{G}_{a,n}^2(x,y_1,\bar{y}) = \tilde{\sigma}_1^n(C_{a,n} + M_{a,n}^{-1}\bar{y}) + \tilde{R}_{a,n}^2(x,y_1,\bar{y}),$$

and therefore locally invariant for $\Phi_{a,n}$.

Now we will study the differentiability of $h_{a,n}$. Let $\Xi_{a,n}$ be the space of bounded continuos maps $L : \mathbf{R}^s \times \mathbf{R} \to \mathcal{L}\left(\mathbf{R}^s \times \mathbf{R}, \mathbf{R}^{u-1}\right)$, $\Xi_{a,n}$ is complete with

$$\|L - L'\|_0 = sup_{x,y_1}\|L(x,y_1) - L'(x,y_1)\|.$$

For every $h \in \Theta_{a,n}$ we define the operator Γ_h in $\Xi_{a,n}$ by

$$\begin{aligned}\Gamma_h(L)(x,y_1) &= \tilde{\sigma}_1^{-n} M_{a,n}^{-1}[L(\tilde{H}_{a,n}(x,y_1,h(x,y_1)), \tilde{G}^1_{a,n}(x,y_1,h(x,y_1))) \\ &\quad \circ(D\tilde{H}_{a,n}(x,y_1,h(x,y_1)), D\tilde{G}^1_{a,n}(x,y_1,h(x,y_1))) \\ &\quad -D\tilde{R}^2_{a,n}(x,y_1,h(x,y_1))] \circ (I_s, 1, L(x,y_1)).\end{aligned} \tag{13}$$

As $D\tilde{H}_{a,n}$, $D\tilde{G}^1_{a,n}$ and $\tilde{\sigma}_1^{-n} D\tilde{R}^2_{a,n}$ are bounded and

$$(\Gamma_h(L) - \Gamma_h(L')) =$$

$$\tilde{\sigma}_1^{-n} M_{a,n}^{-1}[L(\tilde{H}_{a,n}(x,y_1,h(x,y_1)), \tilde{G}^1_{a,n}(x,y_1,h(x,y_1)))$$
$$\circ(D\tilde{H}_{a,n}(x,y_1,h(x,y_1)), D\tilde{G}^1_{a,n}(x,y_1,h(x,y_1)))$$
$$\circ(0^s, 0, (L-L')(x,y_1)) + (L-L')(H_{a,n}(x,y_1,h(x,y_1)), G^1_{a,n}(x,y_1,h(x,y_1)))$$
$$\circ(D\tilde{H}_{a,n}(x,y_1,h(x,y_1)), D\tilde{G}^1_{a,n}(x,y_1,h(x,y_1))) \circ (I_s, 1, L'(x,y_1))$$
$$-D\tilde{R}^2_{a,n}(x,y_1,h(x,y_1)) \circ (0^s, 0, (L-L')(x,y_1))],$$

then, for some sequence k_n, with $k_n \to 0$ when $n \to \infty$,

$$\|\Gamma_h(L) - \Gamma_h(L')\|_0 \leq k_n\|L - L'\|_0 \text{ , for } L, L' \in \Xi_{a,n}.$$

Therefore, for n big enough Γ_h is a contraction. So, the operador $\Delta_{a,n}$ acting on $\Theta_{a.n} \times \Xi_{a,n}$ and defined by $\Delta_{a,n}(h,L) = (\Gamma_{a,n}(h), \Gamma_h(L))$, is continuos and contracts both the base and the fiber. Next, through the Fiber Contraction Theorem, see [HPS], for any n big enough and $a \in J$, $\Delta_{a,n}$ has an unique attractor fixed point $(h_{a,n}, L_{a,n}) \in \Theta_{a.n} \times \Xi_{a,n}$.

We observe that if $h \in \Theta_{a.n}$ is differentiable, then $D\Gamma_{a,n}(h) = \Gamma_h(Dh)$; in general for each $m > 0$, $D\Gamma^m_{a,n}(h) = \Gamma^m_h(Dh)$, as a consequence, of

$$(\Gamma^m_{a,n}(h), \Gamma^m_h(Dh)) \to (h_{a,n}, L_{a,n}) \text{ when } n \to \infty,$$

we conclude that $h_{a,n}$ is differentiable and $Dh_{a,n} = L_{a,n}$.

Let us see that $h_{a,n}$ is, in fact, C^k. For this we will show that $\tilde{S}_{a,n}$ defined by the global graph of $h_{a,n}$ is k-normally hyperbolic (k-normally expanding in this case), then we use

Theorem ([HPS]) *Let $S \subset M$ be an invariant submanifold of $\varphi \in Diff^k(M)$ which is r-normally hyperbolic, $1 \leq r \leq k$. If S is compact and C^1, then is C^r.*

We require $\tilde{S}_{a,n}$ to be compact for using directly the previous result, but the same argument is valid because the maps $\tilde{H}_{a,n}$, $\tilde{G}^1_{a,n}$ and $\tilde{\sigma}_1^{-n}\tilde{R}^2_{a,n}$ have bounded derivatives until order k. (See also Appendix I of [PT])

We recall the definition of r-normally expanding.

Definition . *Given $\varphi \in Diff^k(M), k \geq 1$, and $S \subset M$ a submanifold φ-invariant. S it is said r-normally expanding, $1 \leq r \leq k$, if there is a splitting $T_xM = T_xS \oplus E^u_x$ for each $x \in S$ such that*

• *the splitting depends continuosly on x and $D\varphi(x)(E^u_x) = E^u_{\varphi(x)}$ for each $x \in S$;*

• *for some Riemannian metric, and constants $C > 0$ and $\sigma > 1$ one has for every pair of unitary vectors $u \in T_xS$ and $v \in E^u_x$ and any $n > 0$:*

$$\frac{\|D\varphi^n(x)v\|}{\|D\varphi^n(x)u\|^r} \geq C\sigma^n.$$

Lemma 2.2. *For n big enough and $a \in J$, $\tilde{S}_{a,n}$ is k-normally expanding.*

Proof. In $\mathbf{R}^s \times \mathbf{R} \times \mathbf{R}^{u-1}$ we consider $E_1 = \mathbf{R}^s \times \mathbf{R} \times \{0^{u-1}\}$ and E_2 the orthogonal complement of E_1. For every $s \in \tilde{S}_{a,n}$ we define the cone

$$C_{a,n}(s) = \{v = v_1 + v_2 \in E_1 \oplus E_2 : \|v_1\| \leq |\sigma_u|^{-\frac{n}{2}}\|v_2\|\};$$

we claim that $\{C_{a,n}(s)\}_s$ defines an unstable cone field related to $\tilde{\Phi}_{a,n}$ in $\tilde{S}_{a,n}$, this means:

• $\overline{D\tilde{\Phi}_{a,n}(s)(C_{a,n}(s))} \subset int\left(C_{a,n}(\tilde{\Phi}_{a,n}(s))\right) \cup \{0\}$;

• $\exists \rho > 1$ such that, for each $v = v_1 + v_2 \in C_{a,n}(s)$

$$\|D\tilde{\Phi}_{a,n}(s)v\| \geq \rho\|v\| \text{ ,where } \|v\| = \max(\|v_1\|, \|v_2\|).$$

In fact, given that $D\tilde{H}_{a,n}$ and $D\tilde{G}^1_{a,n}$ are C^k bounded, there exists a constant $K_2 > 0$ such that, for every $v = v_1 + v_2 \in C_{a,n}(s)$

$$\max(\|D\tilde{H}_{a,n}(s)v\|, \|D\tilde{G}^1_{a,n}(s)v\|) \leq K_2\|v_2\|. \tag{14}$$

On the other hand, as $\tilde{G}^2_{a,n}(x, y_1, \bar{y}) = \tilde{\sigma}^n_u(C_{a,n} + M_{a,n}\bar{y}) + \tilde{R}^2_{a,n}(x, y_1, \bar{y})$, then

$$\begin{aligned} \|D\tilde{G}^2_{a,n}(s)v\| \;\geq\; & |\sigma_u|^n\|v_2\|[\|M_{a,n}\| - |\sigma_u|^{-n}\|\partial_{\bar{y}}\tilde{R}^2_{a,n}(s)\| \\ & -|\sigma_u|^{-\frac{3}{2}n}\|(\|\partial_x\tilde{R}^2_{a,n}(s)\| + \|\partial_{y_1}\tilde{R}^2_{a,n}(s)\|\|)] \end{aligned}$$

Therefore, for n big enough and $a \in J$,

$$\|D\tilde{G}^2_{a,n}(s)v\| \geq \frac{1}{2}\inf_{a,n}\|M_{a,n}\||\sigma_u|^n\|v_2\|. \tag{15}$$

Our assertion comes from (14) and (15).

Now, through intersections of the forward image of the cone field along the orbits, it is obtained for each $s \in \tilde{S}_{a,n}$ a $(u-1)$-plane $E^u_{a,n} \subset C_{a,n}(s)$, which varies continuosly on s and

$$\|D\tilde{\Phi}_{a,n} \mid E^u_{a,n}\| \geq \rho_n,$$

for certain ρ_n of the order of $|\sigma_u|^n$. In particular, for each $v \in E^u_{a,n}$ and every $m > 0$,

$$\|D\tilde{\Phi}^m_{a,n}(s)v\| \geq \rho_n\|v\|. \tag{16}$$

Moreover, the bundle $E^u_{a,n}$ converges, when $n \to \infty$, to the canonical bundle of the $(u-1)$-planes $\{0^s\} \times \{0\} \times \mathbf{R}^{u-1}$.

Let us see how is the behavior of $D\tilde{\Phi}_{a,n}$ on $T\tilde{S}_{a,n}$. Let $h_{a,n}$ be the solution of (11); any vector $v \in T_s\tilde{S}_{a,n}$ with $s = (x, y_1, h_{a,n}(x, y_1))$ is of the share

$$v = (w, Dh_{a,n}(x, y_1)w), \;\; w \in \mathbf{R}^s \times \mathbf{R}.$$

As $\|Dh_{a,n}\| \leq \gamma_{a,n}$ then, $\|v\| = \|w\|$; in particular the tangent bundle of $\tilde{S}_{a,n}$ converges to the canonical of the $(s+1)$-planes $\mathbf{R}^s \times \mathbf{R} \times \{0\}^{u-1}$. Of

$$D\tilde{\Phi}_{a,n}(s) = (D\tilde{H}_{a,n}(s), D\tilde{G}^1_{a,n}(s), Dh_{a,n}(\tilde{H}_{a,n}(s), \tilde{G}^1_{a,n}(s)),$$

we have for some constant $K_3 > 0$ and any vector $v \in T_s\tilde{S}_{a,n}$,

$$\|D\tilde{\Phi}_{a,n}(s)v\| \leq K_3\|v\|.$$

So, from this last inequality and (16) follows the lemma. □

Until now we have for each $a \in J$ and n sufficiently large that $\Phi_{a,n}$ admits a locally invariant submanifold of dimension $s+1$ (local graph of $h_{a,n}$) which is C^k and with a normal bundle along which the derivative grows at a rate of $|\sigma_u|$ order.

Finally, let us see that $\Phi_{a,n}$ restristed to

$$S_{a,n} = graph(h_{a,n} \mid [-T,T]^s \text{x} [-T,T])$$

is, unless a coordinates change, C^k-close to the quadratic family

$$\Phi_a(x, y_1) = (0^s, 1 - a y_1^2).$$

In the parametric equations of $S_{a,n}$ given by $h_{a,n}$, for $(x, y_1) \in S_{a,n} \cap \Phi_{a,n}^{-1}(S_{a,n})$ we have

$$\Phi_{a,n}(x, y_1) = (\bar{H}_{a.n}(x, y_1, h_{a,n}(x, y_1)), G_{a,n}^1(x, y_1, h_{a,n}(x, y_1)))$$

then, from Lemma 2.1 follows

$$G_{a,n}^1(x, y_1, h_{a,n}(x, y_1)) = 1 + \frac{c_{a,n}}{2a} - a\left(y_1 - \frac{c_{a,n}}{2a}\right)^2 + r_{a,n}^1(x, y_1, h_{a,n}(x, y_1));$$

where

$$c_{a,n} = -\ell_{a,n}^2(M_{a,n}^{-1}C_{a,n}), \text{ and}$$

$$r_{a,n}^1(x, y_1, h_{a,n}(x, y_1)) = R_{a,n}^1(x, y_1, h_{a,n}(x, y_1)) + q_{a,n}(x, y_1, h_{a,n}(x, y_1))$$

$$-q_{a,n}(M_{a,n}^{-1}C_{a,n}) + \ell_{a,n}^1(h_{a,n}(x, y_1) + M_{a,n}^{-1}C_{a,n}) + \ell_{a,n}^2(h_{a,n}(x, y_1) + M_{a,n}^{-1}C_{a,n}).$$

Clearly $r_{a,n}^1$ converges to zero when $n \to \infty$ in the C^k topology on compact parts, because $\|h_{a,n} + M_{a,n}^{-1}C_{a,n}\|_k \to 0$ when $n \to \infty$ uniformly on compacta.

In this way, in the a-dependent coordinates

$$\begin{cases} \tilde{x} = x \\ \tilde{y}_1 = y_1 - \frac{c_{a,n}}{2a} \end{cases}$$

$\Phi_{a,n}$ restricted to $S_{a,n}$ is given by

$$\Phi_{a,n}(\tilde{x}, \tilde{y}_1) = (\bar{H}_{a,n}(\tilde{x}, \tilde{y}_1, h_{a,n}(\tilde{x}, \tilde{y}_1)), 1 - a\tilde{y}_1^2 + r_{a,n}^1(\tilde{x}, \tilde{y}_1, h_{a,n}(\tilde{x}, \tilde{y}_1)),$$

Now the proof of the theorem is complete. □

Remark 2.1 As $\Phi_{a,n}$ is C^k-close of Φ_a, for a near 2 and n big enough we have (see [PT], [PV]) for the initial family $\{\varphi_\mu\}_\mu$ the following properties:

• There is a sequence of intervals I_n near of $\mu = 0$ such that, for each $\mu \in I_n$, φ_μ has a hyperbolic periodic point P_μ of period $n + N$, of index s and with a one-dimensional weak expansion in its unstable manifold.

• There is a sequence of values $\mu_n \in I_n$ with $\mu_n \to 0$ when $n \to \infty$ such that, the stable and unstable manifolds of P_{μ_n} have an orbit of tangency. In fact, in a local center-stable manifold of P_{μ_n}, $W^{cs}_{loc}(P_{\mu_n})$, there exists a homoclinic tangency of codimension one and sectionally dissipative associated to P_{μ_n}. In addition, due to the continuos dependency of the center-stable manifolds, for n sufficiently large, the family $\{\varphi_\mu\}_{\mu \in I_n}$, when restricted to $W^{cs}_{loc}(P_\mu)$ unfolds generically the tangency associated to the periodic point P_{μ_n}.

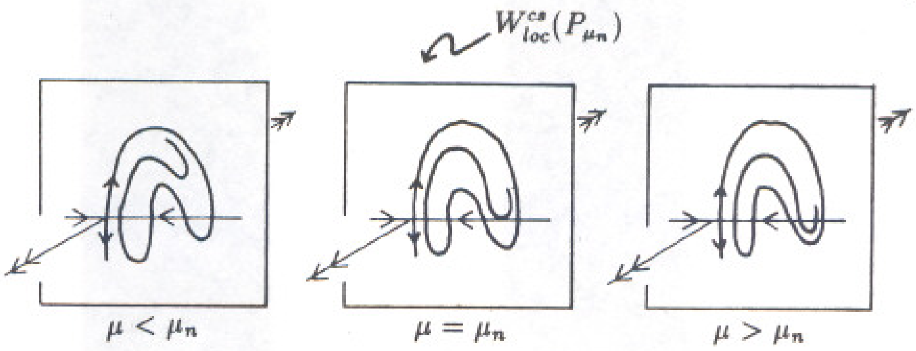

Figure 2.2

As a consequence of our reduction of codimension in homoclinic tangencies together the results of Alligood and Yorke, see [YA] and [PT], we obtain a generalization of the existence of cascades of period doubling bifurcations; which was first proved, recently, by Martín Rivas.

3 Proof of Theorems A and B.

This section will be destined to the proofs of the extension, based in the corresponding results in [PV], of the phenomenon of persistence of homoclinic tangencies.

Since the center-stable manifolds are not, in general, C^∞, we are obliged to remove the hypothesis on linearizing coordinates considered by Palis-Viana. To do that, we use the so called Sil'nikov variables.

3.1 Sil'nikov variables and Renormalizations

Let us define the Sil'nikov variables and state some fundamental properties; the proofs of the known results are basically in section 3 of [AS].

Let $f : U \subset \mathbf{R}^m \to \mathbf{R}^m$ be a C^k diffeomorphism, $k \geq 2$, having the origen as a hyperbolic fixed (or periodic) point. Without generality loss we suppose that f is written in the form

$$f(x,y) = (Ax + f_s(x,y), By + f_u(x,y)) \ ,$$

where $(x,y) \in \mathbf{R}^s \times \mathbf{R}^u$, $s + u = m$; A and B are linear operator with

$$\|A\| \leq \lambda \text{ and } \|B^{-1}\| \leq \sigma^{-1},$$

where $0 < \lambda < 1 < \sigma$, and f_s, f_u are C^k maps vanishe with its derivatives at the origen. We can assume that f_s and f_u are of the order of $(\|x\| + \|y\|)\|x\|$ and $(\|x\|+\|y\|)\|y\|$, respectively; and also $W^s_{loc}(0) \subset (\mathbf{R}^s \times \{0^u\}) \cap U$, $W^u_{loc}(0) \subset (\{0^s\} \times \mathbf{R}^u) \cap U$.

For each integer $n > 0$ and $(x_0, y_n) \in U$ we consider the system

$$\begin{aligned} x_j &= A^j x_0 + \sum_{\ell=1}^{j} A^{j-\ell} f_s(x_{\ell-1}, y_{\ell-1}),\ j = 1, \ldots, n \\ y_j &= B^{j-n} y_n - \sum_{\ell=j+1}^{n} B^{j-\ell} f_u(x_{\ell-1}, y_{\ell-1}),\ j = 0, \ldots, n-1 \end{aligned} \tag{17}$$

Clearly, any solution of this system, $\zeta = \{(x_j, y_j)\}_{j=0}^n$, is a piece of the orbit that at the beginnig is in the u-plane $x = x_0$, and after n iterates is in the s-plane $y = y_n$. The existence of such solutions are garanted by the following

Lemma 3.1. *There are constants $r_0 > 0$ and $n_0 \geq 1$ such that for each $n \geq n_0$, $0 < r < r_0$ and $(x_0, y_n) \in U$ with $\max(\|x_0\| , \|y_n\|) \leq \frac{r}{2}$, the system (17) has a unique solution.* □

Fix points $p = (\xi, 0^u) \in W^s_{loc}(0)$ and $q = (0^s, \eta) \in W^u_{loc}(0)$, we denote by

$$U_p = \{(x,y) : \|x - \xi\| \leq \epsilon_0 \ , \|y\| \leq \epsilon_0\} \text{ , and}$$

$$U_q = \{(x,y) : \|x\| \leq \epsilon_1 \ , \|y - \eta\| \leq \epsilon_1\};$$

we take ϵ_0 and ϵ_1 small enough for $fU_p \cap U_q = \emptyset$, $f^{-1}U_q \cap U_p = \emptyset$ and such that, for every

$$(x_0, y_n) \in V_0 = \{(x,y) : \|x - \xi\| \leq \epsilon_0, \|y - \eta\| \leq \epsilon_1\},$$

(17) admits solution for any $n \geq n_0$. In V_0 are defined maps

$$\Psi^s_n : V_0 \to U_p \text{ and } \Psi^u_n : V_0 \to U_q \text{ given by}$$

$$\Psi^s_n(x,y) = (x, y_0(x,y;n)) \text{ and } \Psi^u_n(x,y) = (x_n(x,y;n), y);$$

where $x_n(x,y;n)$ and $y_0(x,y;n)$ satisfies

$$f^n(x, y_0(x,y;n)) = (x_n(x,y;n), y)$$

This means that $\zeta(x,y;n) = \{f^j(x, y_0(x,y;n))\}_{j=0}^n$ is the solution of (17) with data $(x,y;n)$

We denote by $(x_j(x,y;n), y_j(x,y;n))$, $j = 0, \dots, n$ each point of the solution of (17); obviously $x_0(x,y;n) = x$ and $y_n(x,y;n) = y$.

The values $x_n(x,y;n)$ and $y_0(x,y;n)$ will be called *Sil'nikov variables with data $(x,y;n)$*.

Lemma 3.2. *For n big enough and ϵ_0, ϵ_1 small the maps Φ^s_n and Φ^u_n are diffeomorphisms of class C^k, and for a constant $C > 0$ and $j = 0, \dots, n$ hold the inequalities*

$$\|x_j(.,.;n)\|_k \leq C\lambda^j \text{ and } \|y_j(.,.;n)\|_k \leq C\sigma^{j-n}. \ \square \tag{18}$$

The results stated previously have an inmediate extension for one-parameter family of diffeomorphisms of class C^k. Consider a family $f_\mu : U \subset \mathbf{R}^m \to \mathbf{R}^m$ of class C^k, that is, $(\mu, x) \to (\mu, f_\mu(x, y))$ is C^k. We assume that the origin is a hyperbolic fixed (or periodic) point and

$$f_\mu(x, y) = (A_\mu x + f_{\mu,s}(x, y), B_\mu y + f_{\mu,u}(x, y)),$$

where $(x, y) \in \mathbf{R}^s \times \mathbf{R}^u$, $s + u = m$;

$$Df_\mu(0^s, 0^u) = \begin{bmatrix} A_\mu & 0 \\ 0 & B_\mu \end{bmatrix},$$

for $0 < \lambda = \lambda(\mu) < 1 < \sigma = \sigma(\mu)$:

$$\|A_\mu\| \leq \lambda \text{ and } \|B_\mu^{-1}\| \leq \sigma^{-1}, \text{ and}$$

$$W^s_{loc}(0) \subset (\mathbf{R}^s \times \{0^u\}) \cap U \text{ and } W^u_{loc}(0) \subset (\{0^s\} \times \mathbf{R}^u) \cap U.$$

Like problem (17), we would have the corresponding for the family $\{f_\mu\}_\mu$. If we denote the solution of such problem by

$$\zeta_\mu(x, y; n) = \{(x_{\mu,j}(x, y; n), y_{\mu,j}(x, y; n))\}_{j=0}^{n}$$

then, for n big enough and every $j = 0, \ldots, n$, $x_{\mu,j}$ and $y_{\mu,j}$ depend continuosly with the parameter. Also, for a constant $C > 0$ and $j = 0, \ldots, n$

$$\|x_{\mu,j}(.,.;n)\|_k \leq C\lambda^j \text{ and } \|y_{\mu,j}(.,.;n)\|_k \leq C\sigma^{j-n}. \tag{19}$$

Let $\varphi_\mu : M \to M$ be a family of diffeomorphisms of class C^k, $k \geq 4$, unfolding generically a homoclinic tangency of codimension one in $\mu = 0$. Let $p_0 \in M$ be the fixed (or periodic) point associated to the tangency; assume that the eigenvalues of $D\varphi_0(p_0)$, $\lambda_1, \ldots, \lambda_{m-1}, \sigma$, satisfy

$$0 < |\lambda_{m-1}| \leq \cdots \leq |\lambda_1| < 1 < |\sigma| \text{ and } |\lambda_1 \sigma| < 1$$

and $m = dimM$.

For such families we will prove the following renormalization result without the hypothesis of linearization.

Theorem D. *Let $\{\varphi_\mu\}_\mu$ as above with q a point on the orbit of tangency in $\mu = 0$. There are constant $N > 0$ and a sequence of reparametrizations and μ-dependent coordinates transformations*

$$\varphi_n(a; x, y) = (\mu = \mu(a, n), \varphi_{a,n}(x, y))$$

such that:
i) for compact sets K in $(a; x, y)$-space, $\varphi_n(K) \to (0; q)$ when $n \to \infty$;
ii) the domains of the maps

$$(a; x, y) \to (a; \Phi_{a,n}(x, y) = \varphi_{a,n}^{-1} \circ \varphi_\mu^{n+N} \circ \varphi_{a,n}(x, y))$$

converge, when $n \to \infty$, to $\mathbf{R}_+ \times \mathbf{R}^m$; and to each $a \in J$, compact interval, $\Phi_{a,n}$ converges in the C^{k-4} topology (uniformly compacta) to the quadratic map $\Phi_a(x, y) = (0^{m-1}, 1 - ay^2)$.

In the proof of Theorem D we will use a special μ-dependent coordinates transformations, which allow us to eliminate some "resonant" terms in the local expression of the diffeomorphism in a neighbourhoods of the fixed point.

Lemma 3.3. *Let $\varphi_\mu : U \subset \mathbf{R}^s \times \mathbf{R} \to \mathbf{R}^s \times \mathbf{R}$ be a family of diffeomorphisms of class C^k, $k \geq 4$, having the origin as hyperbolic saddle of index s. There is a μ-dependent change of coordinates of class C^{k-1} in which the family has the local expression*

$$\varphi_\mu(x, y) = (A_\mu \xi + f_\mu(\xi, \eta), \sigma_\mu \eta + \eta^2 \langle g_\mu(\xi, \eta), \xi \rangle); \tag{20}$$

with
i) A_μ is a linear operator in $\mathbf{R}^s$ with $\|A_\mu\| \leq \lambda$, with $0 < \lambda < 1$ some constant; and $\sigma_\mu = \sigma > 1$;
ii) f_μ is C^{k-1} where

$$Df_\mu(0^s, 0),\ f_\mu(0^s, \eta) \text{ and } \frac{\partial^j f_\mu}{\partial \eta^j}(0^s, \eta) \text{ for } j = 1, \ldots, k-1$$

are null; and
iii) g_μ is C^{k-4} and $g_\mu(0^s, 0) = 0$.

Proof. Cf. Lemma 1 in [GS]. □

Proof of Theorem D. Let (ξ, η) be the μ-dependent coodinates of class C^{k-1} in which the family is as (20).

Let $r = (\xi_0, 0) \in W^s_{loc}(p_0) \cap V$ and $q = (0^s, \eta_0) \in W^u_{loc}(p_0) \cap V$ be points on the orbit of tangency, V is the domain of the coordinates; let $N \geq 1$ be an integer such that $\varphi^N_\mu(q) = r$. Then, for μ, ξ and η small enough $\varphi_\mu(\xi, \eta + \eta_0) \in V$; therefore, unless otherwise reparametrizations and linear change of coordinates,

$$\varphi^N_\mu(\xi, \eta + \eta_0) = (q + B\eta + H_1(\mu, \xi, \eta), \mu + C\xi + c\eta^2 + H_2(\mu, \xi, \eta)),$$

where

$$H_1, \partial_\eta H_1, \partial_\mu H_1, \partial_{\eta\eta} H_1, H_2, \partial_\xi H_2, \partial_\mu H_2, \partial_{\mu\eta} H_2 \text{ and } \partial_{\mu\mu} H_2$$

vanishe at $\mu = 0$, $\xi = 0^s$ and $\eta = 0$; $B \in \mathbf{R}^s - \{0^s\}$, $C \in \mathcal{L}(\mathbf{R}^S, \mathbf{R})$ and $c < 0$ (the case $c > 0$ is similar). In this conditions $\varphi^N_\mu(q)$ is such that the η-coordinate, μ, is a local maximum when is restricted to $W^u(p_\mu)$.

Let $V_0 = \{(\xi, \eta) : \|\xi - \xi_0\| \leq \epsilon_0 \ , \ |\eta - \eta_0| \leq \epsilon_1\}$ be such that for n large, μ, ϵ_0 and ϵ_1 small enough, and $(\xi, \eta) \in V_0$ are defined the μ-dependent Sil'nikov variables $x_{\mu,n}$ and $y_{\mu,0}$.

We define the sequence of rescaling and reparametrizations

$$\Theta_n(a; x, y) = (\mu = \mu(a, n), \Theta_{a,n}(x, y)),$$

where

$$\mu = -\frac{a}{c}\sigma^{-2n} + \eta_0\sigma^{-n} - Cx_{\mu,n}(\xi_0, \eta_0; n) +$$

$$\sum_{\ell=1}^{n} \sigma^{-\ell} y^2_{\mu,\ell-1}(\xi_0, \eta_0; n) \langle g_\mu(x_{\mu,\ell-1}(\xi_0, \eta_0; n), y_{\mu,\ell-1}(\xi_0, \eta_0; n)), x_{\mu,\ell-1}(\xi_0, \eta_0; n) \rangle;$$

and $\Theta_{a,n}(x, y) = (\alpha_n x + \xi_0, \beta_n y + \eta_0)$, being that $\alpha_n = \sigma^{-n}\omega^n$ with $1 < \omega < \min(\sigma, |\lambda_1 \sigma|^{-1})$ and $\beta_n = -\frac{a}{c}\sigma^{-n}$.

It is clear that for $a \in J$ and n large, $\Theta_{a,n}(x, y) \in V_0$. If

$$\varphi_n(a; x, y) = (\mu; \varphi_{a,n}(x, y) = \Psi^u_{\mu,n} \circ \Theta_{a,n}(x, y)); \ \Psi^u_{\mu,n}(x, y) = (x_{\mu,n}(x, y; n), y)$$

then, $\varphi_n(K) \to (0; q)$ when $n \to \infty$ for every compact set K in $(a; x, y)$.

On the other hand, since

$$\varphi_\mu^n = \Psi^u_{\mu,n} \circ \left(\Psi^s_{\mu,n}\right)^{-1} \text{ with } \Psi^s_{\mu,n}(x,y) = (x, y_{\mu,0}(x,y;n))$$

then,

$$\begin{aligned} \Phi_{a,n}(x,y) &= \Theta_{a,n}^{-1} \circ \left(\Psi^s_{\mu,n}\right)^{-1} \circ \varphi_\mu^N \circ \Psi^u_{\mu,n} \circ \Theta_{a,n}(x,y) \\ &= \left(\alpha_n^{-1}(\xi_a(x,y) - \xi_0), \beta_n^{-1}(\eta_a(x,y) - \eta_0)\right), \end{aligned}$$

where

$$\xi_a(x,y) = \xi_0 + \beta_n By + H_1(\mu, x_{\mu,n}(\alpha_n x + \xi_0, \beta_n y + \eta_0; n), \beta_n y); \qquad (21)$$

and η_a is given implicitly by

$$y_{\mu,0}(\xi_a(x,y), \eta_a(x,y); n) = \sigma^{-n}\eta_a(x,y) - G_{\mu,n}(\xi_a(x,y), \eta_a(x,y); n)) \qquad (22)$$

and

$$\begin{aligned} y_{\mu,0}(\xi_a(x,y), \eta_a(x,y); n) = \mu + c\beta_n^2 y^2 + Cx_{\mu,n}(\alpha_n x + \xi_0, \beta_n y + \eta_0; n) \qquad (23) \\ + H_2(\mu, x_{\mu,n}(\alpha_n x + \xi_0, \beta_n y + \eta_0; n), \beta_n y); \end{aligned}$$

with

$$\begin{aligned} G_{\mu,n}(\xi,\eta) &= \sum_{\ell=1}^{n} \sigma^{-\ell} y^2_{\mu,\ell-1}(\xi,\eta;n) \langle g_\mu(x_{\mu,\ell-1}(\xi,\eta;n), y_{\mu,\ell-1}(\xi,\eta;n)), \\ & \quad x_{\mu,\ell-1}(\xi,\eta;n)\rangle; \end{aligned}$$

obviously $G_{\mu,n}$ is C^{k-4}.

As H_1, $\partial_\eta H_1$ and $\partial_\mu H_1$ are anulled at $\mu = 0$, $\xi = 0^s$ and $\eta_0 = 0$, then (19) and (21) imply

$$\|\xi_a(.,.) - \xi_0\|_{k-4} \leq K_1 \sigma^{-n}, \qquad (24)$$

for some constant $K_1 > 0$. Next, $\|\alpha_n^{-1}(\xi_a(.,.) - \xi_0)\|_{k-4}$ converges to zero when $n \to \infty$, uniformly on compacta.

From the definition of μ, (22) and (23) follow:

$$\beta_n^{-1}(\eta_a(x,y)-\eta_0) = 1 - ay^2 + \beta_n^{-1}\sigma^n C(x_{\mu,n}(\alpha_n x+\xi_0, \beta_n y+\eta_0;n)$$
$$-x_{\mu,n}(\xi_0,\eta_0;n)) - \beta_n^{-1}\sigma^n(G_{\mu,n}(\xi_a(x,y),\eta_a(x,y)) - G_{\mu,n}(\xi_0,\eta_0))$$
$$+\beta_n^{-1}\sigma^n H_2(\mu, x_{\mu,n}(\alpha_n x+\xi_0,\beta_n y+\eta_0;n),\beta_n y).$$

Let us that

$$\|\beta_n^{-1}(\eta_a(.,.)-\eta_0)-(1-ay^2)\|_{k-4}\to 0, \text{ when } n\to\infty,$$

uniformly on compacta.

Since $H_2, \partial_\xi H_2, \partial_\eta H_2, \partial_\mu H_2, \partial_{\mu\mu} H_2, \partial_{\mu\eta} H_2,$ and $\partial_{\eta\eta} H_2$ are anulled at $\mu = 0$, $\xi = 0^s$ and $\eta = 0$, $\mu = O(\sigma^{-n})$ and $\sigma^{2n}\lambda_1^n\alpha_n < 1$ then, for any multi-index α in (x,y) with $0\le|\alpha|\le k-4$:

$$\sigma^{2n}\|\partial^\alpha(x_{\mu,n}(\alpha_n x+\xi_0,\beta_n y+\eta_0;n)-x_{\mu,n}(\xi_0,\eta_0;n))\| \text{ and}$$

$$\sigma^{2n}\|\partial^\alpha(H_2(\mu,x_{\mu,n}(\alpha_n x+\xi_0,\beta_n y+\eta_0;n),\beta_n y)\|$$

converge to zero on compact parts when $n\to\infty$.

Let us now to estimate the C^{k-4}-norm of

$$\tilde{G}_{\mu,n}(x,y) = G_{\mu,n}(\xi_a(x,y),\eta_a(x,y)) - G_{\mu,n}(\xi_0,\eta_0).$$

It is easy to verify through an elementary computation that on V_0 we have for some constant $K_2>0$ and any multi-index α as above

$$\|\partial^\alpha G_{\mu,n}(.,.)\| \le K_2\sigma^{-n}.$$

In particular, for some constant $K_3>0$

$$\|\tilde{G}_{\mu,n}\| \le K_3\sigma^{-2n}\min(\|\xi_a(.,.)-\xi_0\|\ ,\ \|\eta_a(.,.)-\eta_0\|).$$

Moreover, from (21), (22) and (23)

$$\|\eta_a(.,.)-\eta_0\| \le K_3\sigma^{-n}.$$

Therefore, from (24)

$$\|\tilde{G}_{\mu,n}\| \le K_4\sigma^{-3n}, \text{ for some constant } K_4>0$$

Let α be any multi-index like before, then

$$\partial^\alpha \tilde{G}_{\mu,n}(x,y) =$$

$$= \sum_{j=1}^{|\alpha|} \sum_{|\beta_1|+|\beta_2|=j} \frac{\partial^j}{\partial\xi^{\beta_1}\partial\eta^{\beta_2}} G_{\mu,n}(\xi_a(x,y),\eta_a(x,y)) h_{\beta_1\beta_2}(x,y);$$

where $h_{\beta_1\beta_2}$ are homogeneous polynomials of degree j in $\partial^\gamma\xi_a(x,y)$ and $\partial^\gamma\eta_a(x,y)$ with $1 \leq |\gamma| \leq |\alpha|$. In fact, each monomial of $h_{\beta_1\beta_2}$ are like $X_{i_1}^{\gamma_1}\cdots X_{i_j}^{\gamma_j}$, where $\sum\gamma_k = \alpha$, $|\beta_1| = \#\{i_k = 1\}$, $|\beta_2| = \# : \{i_k = 1\}$ and

$$X_\ell^\gamma = \begin{cases} \partial^\gamma\xi_a(x,y) & \text{if } \ell = 1, \text{ and} \\ \partial^\gamma\eta_a(x,y) & \text{if } \ell = 2 \end{cases}$$

For every $i = 0,\ldots,n$ we introduce the following notation:

$$x_\mu^i(x,y;n) = x_{\mu,i}(\xi_a(x,y),\eta_a(x,y)), \text{ and}$$

$$y_\mu^i(x,y;n) = y_{\mu,i}(\xi_a(x,y),\eta_a(x,y)).$$

Obviously are satisfied

$$x_\mu^{i+1}(x,y;n) = A_\mu x_\mu^i(x,y;n) + f_\mu(x_\mu^i(x,y;n), y_\mu^i(x,y;n));$$

$$y_\mu^{i+1}(x,y;n) = \sigma y_\mu^i(x,y;n)$$

$$+ \left(y_\mu^i(x,y;n)\right)^2 \langle g_\mu(x_\mu^i(x,y;n), y_\mu^i(x,y;n)), x_\mu^i(x,y;n)\rangle;$$

and $x_\mu^0(x,y;n) = \xi_a(x,y)$, , $y_\mu^n(x,y;n) = \eta_a(x,y)$.

Next it is easy to check, using recurrence and standard fact from calculus, that for every multi-index α as above and some constant $K_5 > 0$:

$$\|\partial^\alpha x_\mu^i(x,y;n)\| \leq K_5\sigma^{-n}, \text{ and}$$

$$\|\partial^\alpha y_\mu^i(x,y;n)\| \leq K_5\sigma^{-2n+i} \text{ for } i = 0,\ldots,n.$$

In particular

$$\|\partial^\alpha\eta_a(x,y)\| \leq K_5\sigma^{-n}.$$

Hence, $\|\tilde{G}_{\mu,n}(.,.)\|_{k-4} \leq K_6\sigma^{-3n}$, for is some positive constant K_6.

In this way, we conclude that $\beta_n^{-1}(\eta_a(x,y) - \eta_0)$ converges in the C^{k-4} topology (uniformly on compacta) to the quadratic family $h_a(y) = 1 - ay^2$. So, the proof of the theorem is complete. □

3.2 Persistence of Tangencies.

We are now in a position of extending the results about the abundance, from the topological point of view, of the diffeomorphisms displaying infinitely many coexisting sinks (codimension one case) and the parametrized version of this theorem. Such extensions are the contents of Theorems A and B as stated in the introduction.

Proof of Theorem A. We start considering C^∞ small perturbations of φ such that the perturbed, ϕ, has a hyperbolic fixed point P whose eigenvalues are like (1), satisfy the conditions for C^6 linearizibility and the tangency between $W^s(P)$ and $W^u(P)$ is non-degenerate. Let $\{\phi_\mu\}_\mu$ be any one-parameter family with $\phi_0 = \phi$ and satisfying the generic conditions of Theorem C. From Remark 2.1 we have a sequence of intervals I_n near $\mu = 0$ and values $\mu_n \in I_n$ with $\mu_n \to 0$ when $n \to \infty$ such that, ϕ_{μ_n} has a hyperbolic periodic point P_{μ_n} of period $n + N$, for some $N \geq 1$, with index s and having an one-dimensional weak expansion in $W^u(P_{\mu_n})$. Besides, the stable and unstable manifolds of P_{μ_n} have an orbit of tangency which is parabolic. Associated to P_{μ_n} there is a codimension $u-1$ local center-stable manifold of class C^6 and normally expansive, $u + s = dimM$, such that, in $W^{cs}_{loc}(P_{\mu_n})$ exists a homoclinic tangency of codimension one and sectionally dissipative related to P_{μ_n}. Due to the continuos variation of the center-stable manifold, for n large, the family $\{\phi_\mu\}_{\mu\in I_n}$ when resctricted to $W^{cs}_{loc}(P_\mu)$ unfolds generically this homoclinic tangency.

Let $\mathcal{U}_n$ be a neighbourhood of ϕ_{μ_n} in which the analytic continuation of P_{μ_n} and $W^{cs}_{loc}(P_{\mu_n})$ are defined. For every $\Theta \in \mathcal{U}_n$ we have a map $h_\Theta : Q \subset \mathbf{R}^s \times \mathbf{R} \to \mathbf{R}^{u-1}$, Q an open set, such that tha graph of h_Θ is $W^{cs}_{loc}(P_\Theta)$; these maps vary continuosly on Θ. We denote by $\tilde{\Theta}$ the representation of Θ^{n+N} in the coordinates of $W^{cs}_{loc}(P_\Theta)$ given by h_Θ. So, the family $\{\tilde{\phi}_\mu\}_{\mu\in I_n}$ unfolds generically the sectionally dissipative homoclinic tangency of codimension one in $\mu = \mu_n$ associated to the fixed point P_{μ_n}. Then, there is an open set of tangencies $\tilde{\mathcal{U}}_n$ arbitrarily near $\tilde{\phi}_{\mu_n}$, and residual set $\tilde{\mathcal{R}}_n \subset \tilde{\mathcal{U}}_n$ in which every diffeomorphism displays infinitely many sinks. Hence, the open set $\mathcal{V}_n = \{\Theta \in \mathcal{U}_n : \tilde{\Theta} \in \tilde{\mathcal{U}}_n\}$ and the residual set $\mathcal{R}_n = \{\Theta \in \mathcal{V}_n : \tilde{\Theta} \in \tilde{\mathcal{R}}_n\}$ satisfy the conclusions of the theorem. □

Proof of Theorem B. Using the preceding notation, for every n sufficiently large, the family of class C^6, $\{\tilde{\phi}_\mu\}_{\mu \in I_n}$, unfolds generically a sectionally dissipative homoclinic tangency of codimension one. Therefore, from Theorem D and analogous result of Palis-Viana, there are intervals $J_n \subset I_n$ arbitrarily near μ_n and residual sets $N_n \subset J_n$ such that for $\mu \in N_n$, $\tilde{\varphi}_\mu$ has infinitely many sinks. So, the original diffeomorphism φ_μ, $\mu \in I_n$, displays infinitely many coexisting hyperbolic saddle points of index $s+1$. □

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