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# Controllability of the Semilinear Impulsive Beam Equation

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**Abstract**

In this paper we apply Rothe's Fixed Point Theorem to prove the approximate controllability of the following semilinear impulsive beam Equation

$$\begin{cases} \dfrac{\partial^2 y(t,x)}{\partial t^2} &= 2\beta\Delta\dfrac{\partial y(t,x)}{\partial t} - \Delta^2 y(t,x) + u(t,x) + f(t,y,y_t,u), \text{ in } (0,\tau)\times\Omega, \\ y(t,x) &= \Delta y(t,x) = 0, \quad \text{on } (0,\tau)\times\partial\Omega, \\ y(0,x) &= y_0(x), \quad y_t(x) = v_0(x), x\in\Omega, \\ y_t(t_k^+,x) &= y_t(t_k^-,x) + I_k(t,y(t_k,x),y_t(t_k,x),u(t_k,x)), x\in\Omega, \end{cases}$$

in the states space $Z_1 = D(-\Delta)\times L^2(\Omega)$ endowed with the graph norm, where $\beta > 1$, $\Omega$ is a sufficiently regular bounded domain in $I\!R^N$, the distributed control $u$ belongs to $C([0,\tau];L^2(\Omega))$ and $f, I_k \in C([0,\tau]\times\mathbb{R}\times\mathbb{R};\mathbb{R})$, $k = 1,2,3,\dots,p.$, such that

$$\begin{aligned} |f(t,y,v,u)| &\leq a_0(|y|^{\alpha_0} + |v|^{\alpha_0}) + b_0|u|^{\beta_0} + c_0, \qquad u,y,v\in\mathbb{R}. \\ |I_k(t,y,v,u)| &\leq a_k(|y|^{\alpha_k} + |v|^{\alpha_k}) + b_k|u|^{\beta_k} + c_k, \ k = 1,2,3,\dots,p, \quad u,y,v\in\mathbb{R}. \end{aligned}$$

with $\frac{1}{2} \leq \alpha_k < 1$, $\frac{1}{2} \leq \beta_k < 1$, $k = 0,1,2,3,\dots,p.$
Under this condition we prove the following statement: For all $\tau > 0$ this system is approximately controllable on $[0,\tau]$. Moreover, we could exhibit a sequence of controls steering the nonlinear system from an initial state $z_0$ to an $\epsilon$ neighborhood of the final state $z_1$ at time $\tau > 0$.

*Key words:* semilinear impulsive beam equation, approximate controllability, Rothe's fixed point Theorem
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## 1. Introduction

Many evolutionary processes in nature are characterized by the fact that at certain moments of time they experience an abrupt change, this behavior is observed in several real-life problems: mechanical, chemotherapy, population dynamics, optimal control, ecological, industrial robotics, biotechnology, spread diseases, crops, medical models, diffusive processes, etc. The Theory of Impulsive Differential Equations provides a natural framework to mathematically describe these processes, and one can see the books written by D.D. Bainov, V. Lakshmikantham and P.S. Simeonov [2], N.A. Perestyuk and A.M. Samoilenko [19]. Therefore the area of impulsive differential equations has been developing rapidly due to its applications, which has led to a deeper theoretical study of the subject.

The controllability of Impulsive Evolution Equations has been studied recently for several authors, but most them study the exact controllability only, to mention: D.N. Chalishajar([5]), studied the exact controllability of impulsive partial neutral functional differential equations with infinite delay, B. Radhakrishnan and K. Balachandran([21]) studied the exact controllability of semilinear impulsive integrodifferential evolution systems with nonlocal conditions and S. Selvi, M. Mallika Arjunan([22]) studied the exact controllability for impulsive differential systems with finite delay. To our knowledge, there are a few works on approximate controllability of impulsive semilinear evolution equations, to mention: Lizhen Chen and Gang Li([6]) studied the Approximate controllability of impulsive differential equations with nonlocal conditions, using measure of noncompactness and Monch fixed point theorem, and assuming that the nonlinear term $f(t,z)$ does not depend on the control variable.

In this paper we apply Rothe's Fixed Point Theorem to prove the approximate controllability of the following semilinear impulsive beam Equation

$$\begin{cases} \dfrac{\partial^2 y(t,x)}{\partial t^2} &= 2\beta\Delta\dfrac{\partial y(t,x)}{\partial t} - \Delta^2 y(t,x) + u(t,x) + f(t,y,y_t,u), \\ & \quad \text{in } (0,\tau)\times\Omega, \\ y(t,x) &= \Delta y(t,x) = 0, \quad \text{on } (0,\tau)\times\partial\Omega, \\ y(0,x) &= y_0(x), \quad y_t(x) = v_0(x), x\in\Omega, \\ y_t(t_k^+,x) &= y_t(t_k^-,x) + I_k(t,y(t_k,x),y_t(t_k,x),u(t_k,x)), x\in\Omega, \end{cases} \tag{1.1}$$

in the states space $Z_1 = D(-\Delta)\times L_2(\Omega)$ endowed with the graph norm, where $\beta > 1$, $\Omega$ is a sufficiently regular bounded domain in $I\!R^N$, the distributed control $u$ belongs to $C([0,\tau];L_2(\Omega))$ and $f, I_k \in C([0,\tau]\times\mathbb{R}\times\mathbb{R};\mathbb{R})$, $k = 1,2,3,\dots,p.$, such that

$$|f(t,y,v,u)| \le a_0(|y|^{\alpha_0} + |v|^{\alpha_0}) + b_0|u|^{\beta_0} + c_0, \qquad u,y,v\in\mathbb{R}. \tag{1.2}$$

$$|I_k(t,y,v,u)| \le a_k(|y|^{\alpha_k} + |v|^{\alpha_k}) + b_k|u|^{\beta_k} + c_k,\ k = 1,2,3,\dots,p. \tag{1.3}$$

with $\frac{1}{2} \leq \alpha_k < 1$, $\frac{1}{2} \leq \beta_k < 1$, $k = 0, 1, 2, 3, \ldots, p$.

$$\begin{aligned} y(t_k, x) &= y(t_k^+, x) = \lim_{t \to t_k^+} y(t, x), \quad y(t_k^-, x) = \lim_{t \to t_k^-} y(t, x), \\ y_t(t_k, x) &= y_t(t_k^+, x) = \lim_{t \to t_k^+} y_t(t, x), \quad y_t(t_k^-, x) = \lim_{t \to t_k^-} y_t(t, x). \end{aligned}$$

Here the states space is $Z_1 = [H^2(\Omega) \bigcap H_0^1(\Omega)] \times L_2(\Omega) = D(-\Delta) \times L_2(\Omega)$ endowed with the graph norm; that is to say:

$$\|z\|_{Z_1} = \sqrt{\|(-\Delta)y\|_{L^2}^2 + \|v\|_{L^2}^2}, \quad z = \begin{pmatrix} y \\ v \end{pmatrix} \in Z_1,$$

where

$$\|v\|_{L^2} = \sqrt{\int_\Omega \|v(x)\|^2 dx}, \quad \forall v \in L_2(\Omega).$$

In almost all reference on impulsive differential equations the natural space to work in is the Banach space

$$\begin{aligned} PC([0, \tau]; Z_1) \;=\; & \{z : J = [0, \tau] \to Z_1 : z \in C(J'; Z_1), \exists z(t_k^+, \cdot), z(t_k^-, \cdot) \\ & \text{and} \quad z(t_k, \cdot) = z(t_k^+, \cdot)\}, \end{aligned}$$

endowed with the norm

$$\|z\|_0 = \sup_{t \in [0, \tau]} |z(t, \cdot)|_{Z_1},$$

Definition 1.1. *(**Approximate Controllability**) The system (1.1) is said to be approximately controllable on $[0, \tau]$ if for every $z_0$, $z_1 \in Z_1$, $\varepsilon > 0$ there exists $u \in C([0, \tau]; U)(U = Z)$ such that the solution $z(t)$ of (1.1) corresponding to $u$ verifies:*

$$z(0) = z_0 \quad \textit{and} \quad \|z(\tau) - z_1\|_{Z_1} < \varepsilon, \quad (\textit{Fig.2})$$

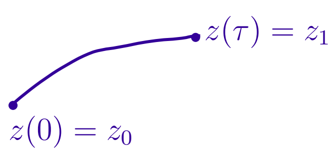


Fig.1

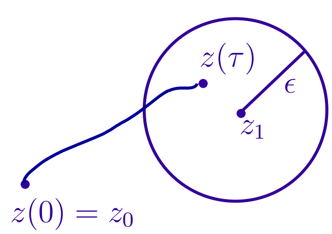


Fig.2

Definition 1.2. (**Controllability to Trajectories**) *The system (1.1) is said to be controllable to trajectories on* $[0, \tau]$ *if for every* $z_0, \hat{z}_0 \in Z_1$ *and* $\hat{u} \in C([0, \tau]; U)$ *there exists* $u \in C([0, \tau]; U)$ *such that the mild solution* $z(t)$ *of (1.1) corresponding to* $u$ *verifies:*

$$z(\tau, z_0, u) = z(\tau, \hat{z}_0, \hat{u}) \quad (Fig.3).$$

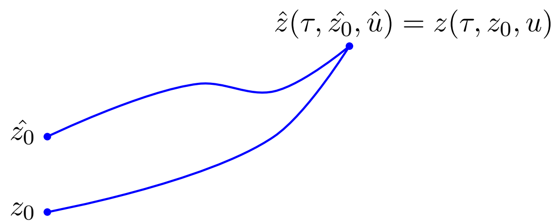


Fig.3

Definition 1.3. (**Null Controllability**) *The system (1.1) is said to be null controllable on* $[0, \tau]$ *if for every* $z_0 \in Z_1$ *there exists* $u \in C([0, \tau]; U)$ *such that the mild solution* $z(t)$ *of (1.1) corresponding to* $u$ *verifies:*

$$z(0) = z_0 \quad and \quad z(\tau) = 0 \quad (Fig.4).$$

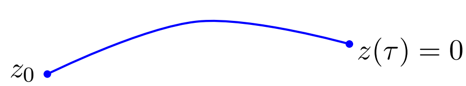


Fig.4

Remark 1.1. *It is clear that exact controllability of the system(1.1) implies approximate controllability, null controllability and controllability to trajectories of the system.But, it is well known ([3]) that due to the diffusion effect or the compactness of the semigroup generated by the linear equation (1.4), the beam equation can never be exactly controllable. We observe also that in the linear case controllability to trajectories and null controllability are equivalent. Nevertheless, the approximate controllability and the null controllability are in general independent. Therefore, in this paper we will be concentrated only on the study of the approximate controllability of the system(1.1).*

Recently, the approximate controllability of the beam equation has been proved in [4]:

$$\begin{cases} \dfrac{\partial^2 y(t,x)}{\partial t^2} & = & 2\beta\Delta\dfrac{\partial y(t,x)}{\partial t} - \Delta^2 y(t,x) + u(t,x) + f(t,y,y_t,u), \\ & & \text{in } (0,\tau)\times\Omega, \\ y(t,x) & = & \Delta y(t,x) = 0, \quad \text{on } (0,\tau)\times\partial\Omega, \\ y(0,x) & = & y_0(x), \quad y_t(x) = v_0(x), x\in\Omega, \end{cases} \tag{1.4}$$

Finally, the approximate controllability of the system (1.1) follows from the approximate controllability of (1.4), the compactness of the semigroup generated by the linear equation, the conditions (1.2) - (1.3) satisfied by the nonlinear terms $f, I_k$ and the following results:

PROPOSITION 1.1. *Let $(X,\Sigma,\mu)$ be a measure space with $\mu(X) < \infty$ and $1 \le q < r < \infty$. Then $L_r(\mu) \subset L_q(\mu)$ and*

$$\|f\|_q \le \mu(X)^{\frac{r-q}{rq}}\|f\|_r, \quad f\in L_r(\mu). \tag{1.5}$$

THEOREM 1.1. *(Rothe's Fixed Theorem, [1],[10], [23]) Let $E$ be a Banach space. Let $B \subset E$ be a closed convex subset such that the zero of $E$ is contained in the interior of $B$. Let $\Phi : B \to E$ be a continuous mapping with $\Phi(B)$ relatively compact in $E$ and $\Phi(\partial B) \subset B$ ($\partial B$ denotes the boundary of $B$). Then there is a point $x^* \in B$ such that $\Phi(x^*) = x^*$.*

## 2. Abstract Formulation of the Problem.

In this section we choose a Hilbert space where system (1.1) can be written as an abstract differential equation; to this end, we consider the following results appearing in [7] pg.46, [9] pg.335 and [11] pg.147:
Let us consider the Hilbert space $Z = L_2(\Omega)$ and $0 < \lambda_1 < \lambda_2 < ... < \lambda_j \longrightarrow \infty$ the eigenvalues of $-\Delta$ with the Dirichlet homogeneous conditions, each one with finite multiplicity $\gamma_j$ equal to the dimension of the corresponding eigenspace. Then we have the following well known properties
(i) There exists a complete orthonormal set $\{\phi_{j,k}\}$ of eigenvectors of $A = -\Delta$.
(ii) For all $z \in D(A)$ we have

$$Az = \sum_{j=1}^{\infty}\lambda_j\sum_{k=1}^{\gamma_j} < \xi, \phi_{j,k} > \phi_{j,k} = \sum_{j=1}^{\infty}\lambda_j E_j z, \tag{2.6}$$

where $< \cdot, \cdot >$ is the inner product in $Z$ and

$$E_j z = \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k}. \tag{2.7}$$

So, $\{E_j\}$ is a family of complete orthogonal projections in $Z$ and $z = \sum_{j=1}^{\infty} E_j z, \quad z \in Z.$

(iii) $-A$ generates an analytic semigroup $\{T(t)\}$ given by

$$T(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j z \quad \text{and} \quad \|T(t)\| \leq e^{-\lambda_1 t}, \quad t \geq 0. \tag{2.8}$$

(iv) The fractional powered spaces $Z^r$ are given by:

$$Z^r = D(A^r) = \{z \in Z : \sum_{j=1}^{\infty} \lambda_j^{2r} \|E_j z\|^2 < \infty\}, \quad r \geq 0,$$

endowed with the norm

$$\|z\|_r = \|A^r z\| = \left\{ \sum_{j=1}^{\infty} \lambda_j^{2r} \|E_j z\|^2 \right\}^{1/2}, \quad z \in Z^r,$$

and

$$A^r z = \sum_{j=1}^{\infty} \lambda_j^r E_j z. \tag{2.9}$$

For $r = 1$ we define $Z_1 = Z^1 \times Z$, which is a Hilbert space with norm given by

$$\left\| \begin{bmatrix} y \\ v \end{bmatrix} \right\|_{Z_1} = \sqrt{\|y\|_1^2 + \|v\|^2}.$$

Hence, (1.1) can be written as an abstract system of ordinary differential equations in the Hilbert space $Z_1$ as follows:

$$\begin{cases} y' = v, \\ \\ v' = -A^2 y - 2\beta A v + u + f(t, y, v, u). \end{cases} \tag{2.10}$$

Finally, the equation (1.1) can be rewritten as a first order system of ordinary differential equations in the Hilbert space $Z_1$ as follows:
Consequently, system (1.1) can be written as an abstract impulsive differential equations in $Z_1$:

$$\begin{cases} z' = \mathcal{A}z + Bu + F(t, z, u), \ z \in Z_1, \quad t \in (0, \tau], t \neq t_k, \\ z(0) = z_0, \\ z(t_k^+) = z(t_k^-) + I_k^e(t_k, z(t_k), u(t_k)), \quad k = 1, 2, 3, \ldots, p. \end{cases} \tag{2.11}$$

where $u \in C([0,\tau];U)$, $U = Z = L_2(\Omega)$,

$$\begin{bmatrix} 0 & I_Z \\ -A^2 & -2\beta A \end{bmatrix} \tag{2.12}$$

is an unbounded linear operator with domain

$$D(\mathcal{A}) = \{y \in H^4(\Omega) : \ y = \Delta y = 0\} \times D(A),$$

i.e.,

$$z = \begin{bmatrix} y \\ v \end{bmatrix} \in D(\mathcal{A}) \Leftrightarrow y \in \{y \in H^4(\Omega) : \ y = \Delta y = 0\} \quad \text{and} \quad v \in D(A).$$

$I = I_Z : Z \to Z = L_2(\Omega)$ is the identity operator, $B : U \to Z_1$, $B = \begin{bmatrix} 0 \\ I_Z \end{bmatrix}$ is a bounded linear operator and $F, I_k^e : [0,\tau] \times Z_1 \times U \longrightarrow Z_1$ are smooth functions defined:

$$F(t,z,u)(x) = \begin{bmatrix} 0 \\ f(t,w(x),v(x),u(x)) \end{bmatrix} \quad \text{and} \quad I_k^e(t,z,u) = \begin{bmatrix} 0 \\ I_k(t,w(x),v(x),u(x)) \end{bmatrix}. \tag{2.13}$$

$\forall x \in \Omega, k = 1,2,\ldots,p.$
On the other hand, from conditions (1.2) and (1.3) we get the following estimate.

Proposition 2.1. *Under the conditions (1.2)-(1.3) the functions $F, I_k^e : [0,\tau] \times Z_1 \times U \to Z_1$,$k = 1,2,3,\ldots,p$, defined above satisfy $\forall u \in Z = L_2(\Omega)$, $\forall z \in Z_1$:*

$$\|F(t,z,u)\|_{Z_1} \leq \bar{a}_0\|z\|_{Z_1}^{\alpha_0} + \bar{b}_0\|u\|_Z^{\beta_0} + \bar{c}_0 \tag{2.14}$$

$$\|I_k^e(t,z,u)\|_{Z_1} \leq \bar{a}_k\|z\|_{Z_1}^{\alpha_k} + \bar{b}_k\|u\|_Z^{\beta_k} + \bar{c}_k, \quad k = 1,2,3,\ldots,p. \tag{2.15}$$

Proof

$$\begin{aligned}
\|F(t,z,u)\|_{Z_1}^2 &= \int_\Omega |f(t,w(x),v(x),u(x))|^2 dx \\
&\leq \int_\Omega \{a_0(|w(x)|^{\alpha_0} + |v(x)|^{\alpha_0}) + b_0|u(x)|^{\beta_0} + c_0\}^2 dx \\
&\leq \int_\Omega \{2^2 a_0^2 \left(|w(x)|^{2\alpha_0} + |v(x)|^{2\alpha_0}\right) + 4^2 b_0^2 |u(x)|^{2\beta_0} + 4^2 c_0^2\} dx \\
&\leq 2^2 a_0^2 \int_\Omega \left(|w(x)|^{2\alpha_0} + |v(x)|^{2\alpha_0}\right) dx + 4^2 b_0^2 \int_\Omega |u(x)|^{2\beta_0} dx + 4^2 c_0^2 \mu(\Omega).
\end{aligned}$$

Then

$$\begin{aligned}\|F(t,z,u)\|_{Z_1} &\leq 2a_0\left(\int_\Omega \left(|w(x)|^{2\alpha_0}+|v(x)|^{2\alpha_0}\right)dx\right)^{\frac{1}{2}}\\ &+ 4b_0\left(\int_\Omega |u(x)|^{2\beta_0}dx\right)^{\frac{1}{2}}+4c_0\sqrt{\mu(\Omega)}\\ &\leq 2a_0\left(\|w\|^{\alpha_0}_{L_{2\alpha_0}}+\|v\|^{\alpha_0}_{L_{2\alpha_0}}\right)+4b_0\|z\|^{\beta_0}_{L_{2\beta_0}}+4c_0\sqrt{\mu(\Omega)}\end{aligned}$$

Now, since $\frac{1}{2}\leq \alpha_0 < 1 \Leftrightarrow 1 \leq 2\alpha_0 < 2$ and $\frac{1}{2}\leq \beta_0 < 1 \Leftrightarrow 1\leq 2\beta_0 < 2$ applying proposition 1.1, we obtain that:

$$\|F(t,z,u)\|_{Z_1}\leq 2a_0\mu(\Omega)^{\frac{1-\alpha_0}{\alpha_0}}\left(\|w\|^{\alpha_0}_Z+\|v\|^{\alpha_0}_Z\right)+4b_0\mu(\Omega)^{\frac{1-\beta_0}{\beta_0}}\|u\|^{\beta_0}_U+4c_0\sqrt{\mu(\Omega)}.$$

Now, from the continuous inclusion $Z^1\subset Z$, there exists a constant $L>0$ such that

$$\|z\|\leq L\|z\|_1,\quad \forall z\in Z^1.$$

Therefore,

$$\|F(t,z,u)\|_{Z_1}\leq 2a_0\mu(\Omega)^{\frac{1-\alpha_0}{\alpha_0}}\left(L\|w\|^{\alpha_0}_1+\|v\|^{\alpha_0}_Z\right)+4b_0\mu(\Omega)^{\frac{1-\beta_0}{\beta_0}}\|u\|^{\beta_0}_U+4c_0\sqrt{\mu(\Omega)}.$$

So, putting $m=\max\{L,1\}$ we obtain that

$$\|F(t,z,u)\|_{Z_1}\leq m2a_0\mu(\Omega)^{\frac{1-\alpha_0}{\alpha_0}}\left(\|w\|^{\alpha_0}_1+\|v\|^{\alpha_0}_Z\right)+4b_0\mu(\Omega)^{\frac{1-\beta_0}{\beta_0}}\|u\|^{\beta_0}_U+4c_0\sqrt{\mu(\Omega)}.$$

Finally, using the fact that $\|w\|_1,\|v\|\leq\sqrt{\|w\|^2_1+\|v\|^2}=\|z\|_{Z_1}$ we obtain that

$$\|F(t,z,u)\|_{Z_1}\leq 4ma_0\mu(\Omega)^{\frac{1-\alpha_0}{\alpha_0}}\|z\|^{\alpha_0}_{Z_1}+4b_0\mu(\Omega)^{\frac{1-\beta_0}{\beta_0}}\|u\|^{\beta_0}_U+4c_0\sqrt{\mu(\Omega)}.$$

Analogously, we obtain the following estimate for $k=1,2,3,\ldots,p$.

$$\|I^e_k(t,z,u)\|_{Z_1}\leq 4ma_k\mu(\Omega)^{\frac{1-\alpha_k}{\alpha_k}}\|z\|^{\alpha_k}_{Z_1}+4b_k\mu(\Omega)^{\frac{1-\beta_k}{\beta_k}}\|u\|^{\beta_k}_U+4c_k\sqrt{\mu(\Omega)},$$

which completes the proof.
Now, using the following Lemma from [13], in [4] it was proved that the linear unbounded operator $\mathcal{A}$ given by the linear equation (2.12) generates a strongly continuous compact semigroup in the space $Z_1$, which decays exponentially to zero.

Theorem 2.1. *The operator $\mathcal{A}$ is the infinitesimal generator of a strongly continuous compact semigroup $\{T(t)\}_{t\geq 0}$ represented by*

$$T(t)z=\sum_{j=1}^{\infty}e^{\mathbb{A}_j t}P_j z,\ z\in Z_1,\ t\geq 0, \tag{2.16}$$

*where* $\{P_j\}_{j\geq 0}$ *is a complete family of orthogonal projections in the Hilbert space* $Z_1$ *given by*

$$P_j = diag(E_j, E_j), \tag{2.17}$$

*and*

$$\mathbb{A}_j = K_j P_j,\ K_j = \begin{bmatrix} 0 & 1 \\ -\lambda_j^2 & -2\beta\lambda_j \end{bmatrix}\ j \geq 1,$$

*and the adjoint operator* $\mathbb{A}^*$ *of the operator* $\mathbb{A}$ *is given by*

$$\mathbb{A}_j^* = \tilde{K}_j P_j,\ \tilde{K}_j = \begin{bmatrix} 0 & -1 \\ \lambda_j^2 & -2\beta\lambda_j \end{bmatrix}\ j \geq 1,$$

*Moreover, the eigenvalues* $\sigma_1(j),\ \sigma_2(j)$*, of the matrix* $K_j$ *are simple and given by:*

$$\sigma_1(j) = -\lambda_j\rho_1,\ \sigma_2(j) = -\lambda_j\rho_2$$

*where* $0 < \rho_1 < \rho_2$ *are given by*

$$\rho_1 = \beta - \sqrt{\beta^2 - 1} \quad \textit{and} \quad \rho_2 = \beta + \sqrt{\beta^2 - 1};\ ,\ \beta^2 > 1,$$

*and this semigroup decays exponentially to zero*

$$\| T(t) \| \leq me^{-\mu t},\ t \geq 0,$$

*where* $\mu = \lambda_1\rho_1$ *and* $\| T(t) \| = \sup_{\|z\|=1} \|T(t)z\|$.

## 3. Controllability of the Linear Equation without Impulses

In this section we shall recall some results about the approximate controllability of the linear system without impulses. To this end, for all $z_0 \in Z_1$ and $u \in L_2([0,\tau];U)$ the initial value problem

$$\begin{cases} z'(t) = \mathcal{A}z(t) + Bu(t), & z \in Z_1, \\ z(0) = z_0, \end{cases} \tag{3.18}$$

admits only one mild solution given by

$$z(t) = T(t)z_0 + \int_0^t T(t-s)Bu(s)ds;\ t \in [0,\tau]. \tag{3.19}$$

Definition 3.1. *For the system (3.18) we define the following concept: The controllability map (for* $\tau > 0$*)* $G : L_2([0,\tau];U) \longrightarrow Z_1$ *is defined by*

$$Gu = \int_0^\tau T(\tau - s)Bu(s)ds, \tag{3.20}$$

*whose adjoint operator* $G^* : Z_1 \longrightarrow L_2([0,\tau];U)$ *is given by*

$$(G^*z)(s) = B^*T(\tau - s)z,\ \forall s \in [0,\tau],\ \forall z \in Z_1. \tag{3.21}$$

The following lemma is trivial

LEMMA 3.1. *The equation (3.18) is approximately controllable on* $[0, \tau]$ *if, and only if,* $\overline{Rang(G)} = Z_1$.

The following Theorem is a characterization of the controllability of the system (3.18):

THEOREM 3.1. *(se [4],[12], [14],[15], [16]) The system (3.18) is approximately controllable on* $[0, \tau]$ *if, and only if, any one of the following conditions hold:*
*a)* $\overline{Rang(G)} = Z_1$.
*b)* $Ker(G^*) = \{0\}$.
*c)* $B^\star T^\star(s)z = 0, \ \forall s \in [0, \tau] \Longrightarrow z = 0$.
*d)* $\lim_{\alpha \to 0^+} \alpha(\alpha I + GG^*)^{-1}z = 0$.
*e)* $\langle GG^* z, z\rangle > 0, \ z \neq 0$ *in* $Z_1$.
*f) For all* $z \in Z_1$ *we have* $Gu_\alpha = z - \alpha(\alpha I + GG^*)^{-1}z$, *where*

$$u_\alpha = G^*(\alpha I + GG^*)^{-1}z, \ \alpha \in (0, 1].$$

*So,* $\lim_{\alpha \longrightarrow 0} Gu_\alpha = z$ *and the error* $E_\alpha z$ *of this approximation is given by*

$$E_\alpha z = \alpha(\alpha I + GG^*)^{-1}z, \ \alpha \in (0, 1].$$

REMARK 3.1. *The Theorem 3.1 implies that the family of linear operators*

$$\Gamma_\alpha z = B^\star T^*(\cdot)(\alpha I + GG^*)^{-1}z = G^*(\alpha I + GG^*)^{-1}z,$$

*is an approximate inverse for the right of the operator* $G$ *in the sense that*

$$\lim_{\alpha \longrightarrow 0} G\Gamma_\alpha = I$$

*in the strong topology.*

PROPOSITION 3.1. *If the* $\overline{Rang(G)} = Z_1$, *then*

$$\sup_{\alpha > 0} \|\alpha(\alpha I + GG^*)^{-1}\| \leq 1.$$

The following result was proved in [4].

THEOREM 3.2. *System (3.18) is approximately controllable on* $[0, \tau]$. *Moreover, a sequence of controls steering the system (3.18) from initial state* $z_0$ *to an* $\epsilon$ *neighborhood of the final state* $z_1$ *at time* $\tau > 0$ *is given by*

$$u_\alpha(t) = B^*_\omega T^*(\tau - t)(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0),$$

*and the error of this approximation* $E_\alpha$ *is given by*

$$E_\alpha = \alpha(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0).$$

## 4. Controllability of the Semilinear System

In this section we shall prove the main result of this paper, the approximate controllability of the Semilinear Impulsive Beam Equation given by (1.1), which is equivalent to prove the approximate controllability of the system (2.11). To this end, for all $z_0 \in Z_1$ and $u \in C([0,\tau];U)$ the initial value problem

$$\begin{cases} z' = \mathcal{A}z + Bu + F(t,z,u), & t \in (0,\tau], t \neq t_k, \quad z \in Z_1 \\ z(0) = z_0, \\ z(t_k^+) = z(t_k^-) + I_k^e(t, z(t_k), u(t_k)), & k = 1,2,3,\dots,p. \end{cases} \tag{4.22}$$

admits only one mild solution given by

$$\begin{aligned} z_u(t) &= T(t)z_0 + \int_0^t T(t-s)Bu(s)ds & (4.23)\\ &+ \int_0^t T(t-s)F(s,z_u(s),u(s))ds & (4.24)\\ &+ \sum_{0<t_k<t} T(t-t_k)I_k^e(t_k, z(t_k), u(t_k)), \quad t \in [0,\tau]. \end{aligned}$$

The main idea of our approach is to transform the problem of controllability in to the existence to a fixed point for the following nonlinear operator:
To this end, we define the operator
$\mathcal{K}^\alpha : PC([0,\tau];Z_1) \times C([0,\tau];U) \to PC([0,\tau];Z_1) \times C([0,\tau];U)$ by the following formula:

$$(y,v) = (\mathcal{K}_1^\alpha(z,u), \mathcal{K}_2^\alpha(z,u)) = \mathcal{K}^\alpha(z,u)$$

where

$$\begin{aligned} y(t) &= \mathcal{K}_1^\alpha(z,u)(t) = T(t)z_0 + \int_0^t T(t-s)B(\Gamma_\alpha \mathcal{L}(z,u))(s)ds & (4.25)\\ &+ \int_0^t T(t-s)F(s,z(s),u(s))ds + \sum_{0<t_k<t} T(t-t_k)I_k^e(t_k, z(t_k), u(t_k)) \end{aligned}$$

and

$$v(t) = \mathcal{K}_2^\alpha(z,u)(t) = (\Gamma_\alpha \mathcal{L}(z,u))(t) = B^*T^*(\tau - t)(\alpha I + \mathcal{W})^{-1}\mathcal{L}(z,u), \tag{4.26}$$

with $\mathcal{L} : PC([0,\tau];Z_1) \times C([0,\tau];U) \to Z_1$ is given by

$$\begin{aligned} \mathcal{L}(z,u) &= z_1 - T(\tau)z_0 - \int_0^\tau T(\tau-s)F(s,z(s),u(s))ds & (4.27)\\ &- \sum_{0<t_k<\tau} T(\tau-t_k)I_k^e(t_k, z(t_k), u(t_k)). \end{aligned}$$

THEOREM 4.1. *The nonlinear system (1.1) is approximately controllable on* $[0, \tau]$. *Moreover, a sequence of controls steering the system (1.1) from initial state* $z_0$ *to an* $\epsilon$*-neighborhood of the final state* $z_1$ *at time* $\tau > 0$ *is given by*

$$u_\alpha(t) = B^* T^*(\tau - t)(\alpha I + \mathcal{W})^{-1} \mathcal{L}(z_\alpha, u_\alpha),$$

*and the error of this approximation* $E_\alpha z$ *is given by*

$$E_\alpha z = \alpha(\alpha I + \mathcal{W})^{-1} \mathcal{L}(z_\alpha, u_\alpha),$$

*where*

$$\begin{aligned} z_\alpha(t) &= T(t)z_0 + \int_0^t T(t-s)Bu_\alpha(s)ds \\ &+ \int_0^t T(t-s)F(s, z_\alpha(s), u_\alpha(s))ds \\ &+ \sum_{0<t_k<t} T(t-t_k)I_k^e(t_k, z_\alpha(t_k), u_\alpha(t_k)), \quad t \in [0, \tau]. \end{aligned}$$

Proof We shall prove this Theorem by claims. Before that, we note that $\|B\| = 1$ and $\|T(t)\| \le me^{-\mu t}, \quad t \ge 0$.

**Claim 1.** The operator $\mathcal{K}^\alpha$ is continuous. In fact, it is enough to prove that the operators:

$$\mathcal{K}_1^\alpha : PC([0,\tau]; Z_1) \times C([0,\tau]; U) \to PC([0,\tau]; Z_1)$$

and

$$\mathcal{K}_2^\alpha : PC([0,\tau]; Z_1) \times C([0,\tau]; U) \to C([0,\tau]; U),$$

define above are continuous. The continuity of $\mathcal{K}_1^\alpha$ follows from the continuity of the nonlinear functions $F(t, z, u)$, $I_k^e(t, z, u)$ and the following estimate

$$\begin{aligned} \|\mathcal{K}_1^\alpha(z,u) - \mathcal{K}_1^\alpha(w,v)\| &\le L_1 \sup_{s\in[0,\tau]} \|F(s, z(s), u(s)) - F(s, w(s), v(s))\| \\ &+ L_2 \sum_{0<t_k<\tau} \|I_k^e(t_k, z(t_k), u(t_k)) - I_k^e(t_k, w(t_k), v(t_k))\|. \end{aligned}$$

where $L_1 = \tau(\tau\|(\alpha I + \mathcal{W})^{-1}\| + 1)$ and $L_2 = (1 + \tau\|(\alpha I + \mathcal{W})^{-1}\|)$.

The continuity of the operator $\mathcal{K}_2^\alpha$ follows from the continuity of the operators $\mathcal{L}$ and $\Gamma_\alpha$ define above.

**Claim 2.** The operator $\mathcal{K}^\alpha$ is compact. In fact, let $D$ be a bounded subset of $PC(J; Z_1) \times C(J; U)$. It follows that $\forall (z, u) \in D$, we have

$$\begin{aligned} \|F(\cdot, z, u)\|_0 \le L_3, &\qquad \|(\alpha I + \mathcal{W})^{-1}\mathcal{L}(z,u)\| \le L_4, \\ \|\mathcal{L}(z,u)\| \le L_5, &\quad \|I_k^e(\cdot, z, u)\|_0 \le l_k, \quad k = 1, 2, \ldots, p. \end{aligned}$$

Therefore , $\mathcal{K}^\alpha(D)$ is uniformly bounded.
Now, consider the following estimate:

$$\begin{aligned}\|\mathcal{K}^\alpha(z,u)(t_2)-\mathcal{K}^\alpha(z,u)(t_1)\|_1 &= \|\mathcal{K}_1^\alpha(z,u)(t_2)-\mathcal{K}_1^\alpha(z,u)(t_1)\| \\ &+ \|\mathcal{K}_2^\alpha(z,u)(t_2)-\mathcal{K}_2^\alpha(z,u)(t_1)\|,\end{aligned}$$

where

$$\begin{aligned}\|\mathcal{K}_1^\alpha(z,u)(t_2)-\mathcal{K}_1^\alpha(z,u)(t_1)\| &\leq \|T(t_2)-T(t_1)\|\|z_0\| \\ &+ \int_0^{t_1}\|T(t_2-s)-T(t_1-s)\|\|\mathcal{L}(z,u)(s)\|ds \\ &+ \int_{t_1}^{t_2}\|T(t_2-s)\|\|\mathcal{L}(z,u)(s)\|ds \\ &+ \int_0^{t_1}\|T(t_2-s)-T(t_1-s)\|\|F(s,z(s),u(s))\|ds \\ &+ \int_{t_1}^{t_2}\|T(t_2-s)\|\|F(s,z(s),u(s))\|ds \\ &+ \sum_{0<t_k<t_1}\|T(t_2-t_k)-T(t_1-t_k)\|\|I_k^e(t_k,z(t_k),u(t_k))\| \\ &+ \sum_{t_1<t_k<t_2}\|T(t_2-t_k)I_k^e(t_k,z(t_k),u(t_k))\|,\end{aligned}$$

and

$$\|\mathcal{K}_2^\alpha(z,u)(t_2)-\mathcal{K}_2^\alpha(z,u)(t_1)\| \leq \|T^*(\tau-t_2)-T^*(\tau-t_1)\|\|(\alpha I+\mathcal{W})^{-1}\mathcal{L}(z,u)\|.$$

On the other hand, since $T(t)$ is a compact operator for $t>0$, then from [18] we know that the function $0<t\to T(t)$ is uniformly continuous. so,

$$\lim_{|t_2-t_1|\to 0}\|T(t_2)-T(t_1)\|=0,\quad t_2,t_1>0.$$

Consequently, if we take a sequence $\{\phi_j : j=1,2,\dots\}$ on $\mathcal{K}^\alpha(D)$, this sequence is uniformly bounded and equicontinuous on the interval $[0,t_1]$ and, by Arzela theorem, there is a subsequence $\{\phi_j^1 : j=1,2,\dots\}$ of $\{\phi_j : j=1,2,\dots\}$, which is uniformly convergent on $[0,t_1]$.
Consider the sequence $\{\phi_j^1 : j=1,2,\dots\}$ on the interval $(t_1,t_2]$. On this interval the sequence $\{\phi_j^1 : j=1,2,\dots\}$ is uniformly bounded and equicontinuous, and for the same reason, it has a subsequence $\{\phi_j^2\}$ uniformly convergent on $[0,t_2]$.
Continuing this process for the intervals $(t_2,t_3]$, $(t_3,t_4]$, $\dots$, $(t_p,\tau]$, we see that the sequence $\{\phi_j^{p+1} : j=1,2,\dots\}$ converges uniformly on the interval $[0,\tau]$. This means

that $\overline{\mathcal{K}^{\alpha}(D)}$ is compact, which implies that the operator $\mathcal{K}^{\alpha}$ is compact.
**Claim 3.**

$$\lim_{|||(z,u)|||\to\infty} \frac{|||\mathcal{K}^{\alpha}(z,u)|||}{|||(z,u)|||} = 0,$$

where $|||(z,u)||| = \|z\|_0 + \|u\|_0$ is the norm in the space $PC([0,\tau];Z) \times C(0,\tau;U)$. In fact, consider the following estimates:

$$\|\mathcal{L}(z,u)\| \leq M_1 + M_2\{\overline{a}_0\|z\|^{\alpha_0} + \overline{b}_0\|u\|^{\beta_0} + \overline{c}_0\} + M_3 \sum_{0<t_k<\tau} \{\overline{a}_k\|z\|^{\alpha_k} + \overline{b}_k\|u\|^{\beta_k} + \overline{c}_k\},$$

where

$$M_1 = \|z_1\| + me^{-\mu\tau}\|z_0\|, \quad M_2 = \frac{m}{-\mu}(e^{-\mu\tau} - 1) \quad \text{and} \quad M_3 = me^{-\mu\tau}.$$

$$\begin{aligned} \|\mathcal{K}_2^{\alpha}(z,u)\| &\leq M_3M_1\|(\alpha I + \mathcal{W})^{-1}\| + M_3M_2\|(\alpha I + \mathcal{W})^{-1}\|\{\overline{a}_0\|z\|^{\alpha_0} + \overline{b}_0\|u\|^{\beta_0} + \overline{c}_0\} \\ &+ M_3M_2\|(\alpha I + \mathcal{W})^{-1}\| \sum_{0<t_k<\tau} \{\overline{a}_k\|z\|^{\alpha_k} + \overline{b}_k\|u\|^{\beta_k} + \overline{c}_k\}. \end{aligned}$$

and

$$\begin{aligned} \|\mathcal{K}_1^{\alpha}(z,u)\| &\leq M_3\{\|z_0\| + M_1M_2\|(\alpha I + \mathcal{W})^{-1}\|\} \\ &+ M_2\{1 + M_2M_3\|(\alpha I + \mathcal{W})^{-1}\|\}\{\overline{a}_0\|z\|^{\alpha_0} + \overline{b}_0\|u\|^{\beta_0} + \overline{c}_0\} \\ &+ M_3\{1 + M_2M_3\|(\alpha I + \mathcal{W})^{-1}\|\} \sum_{0<t_k<\tau} \{\overline{a}_k\|z\|^{\alpha_k} + \overline{b}_k\|u\|^{\beta_k} + \overline{c}_k\}. \end{aligned}$$

Therefore,

$$\begin{aligned} |||\mathcal{K}^{\alpha}(z,u)||| &= \|\mathcal{K}_1^{\alpha}(z,u)\| + \|\mathcal{K}_2^{\alpha}(z,u)\| \leq M_4 \\ &+ \{M_3M_2\|(\alpha I + \mathcal{W})^{-1}\|\{1 + M_2\} + M_2\}\{\overline{a}_0\|z\|^{\alpha_0} + \overline{b}_0\|u\|^{\beta_0} + \overline{c}_0\} \\ &+ \{M_3M_2\|(\alpha I + \mathcal{W})^{-1}\|\{1 + M_3\} + M_3\} \sum_{0<t_k<\tau} \{\overline{a}_k\|z\|^{\alpha_k} + \overline{b}_k\|u\|^{\beta_k} + \overline{c}_k\}, \end{aligned}$$

where $M_4$ is given by:

$$M_4 = M_3\{\|z_0\| + (M_2 + 1)M_1\|(\alpha I + \mathcal{W})^{-1}\|\}.$$

Hence

$$\begin{aligned} \frac{|||\mathcal{K}^{\alpha}(z,u)|||}{|||(z,u)|||} &\leq \frac{M_4}{\|z\| + \|u\|} \\ &+ \{M_3M_2\|(\alpha I + \mathcal{W})^{-1}\|\{1 + M_2\}\} \\ &\quad \times\{\overline{a}_0\|z\|^{\alpha_0 - 1} + \overline{b}_0\|u\|^{\beta_0 - 1} + \frac{\overline{c}_0}{\|z\| + \|u\|}\} \\ &+ \{M_3M_2\|(\alpha I + \mathcal{W})^{-1}\|\{1 + M_3\} + M_3\} \times \\ &\quad \sum_{0<t_k<\tau} \{\overline{a}_k\|z\|^{\alpha_k - 1} + \overline{b}_k\|u\|^{\beta_k - 1} + \frac{\overline{c}_k}{\|z\| + \|u\|}\}, \end{aligned}$$

and

$$\lim_{|||(z,u)|||\to\infty} \frac{|||\mathcal{K}^\alpha(z,u)|||}{|||(z,u)|||} = 0. \tag{4.28}$$

**Claim 4.**The operator $\mathcal{K}^\alpha$ has a fixed point. In fact, for a fixed $0 < \rho < 1$, there exists $R > 0$ big enough such that

$$|||\mathcal{K}^\alpha(z,u)||| \leq \rho|||(z,u)|||, \quad |||(z,u)||| = R.$$

Hence, if we denote by $B(0,R)$ the ball of center zero and radio $R > 0$, we get that $\mathcal{K}^\alpha(\partial B(0,R)) \subset B(0,R)$. Since $\mathcal{K}^\alpha$ is compact and maps the sphere $\partial B(0,R)$ into the interior of the ball $B(0,R)$, we can apply Rothe's fixed point Theorem 1.1 to ensure the existence of a fixed point $(z_\alpha, u_\alpha) \in B(0,R) \subset PC([0,\tau];Z_1) \times C(0,\tau;U)$ such that

$$(z_\alpha, u_\alpha) = \mathcal{K}^\alpha(z_\alpha, u_\alpha). \tag{4.29}$$

**Claim 5.** The sequence $\{(z_\alpha, u_\alpha)\}_{\alpha\in(0,1]}$ is bounded. In fact, for the purpose of contradiction, let us assume that $\{(z_\alpha, u_\alpha)\}_{\alpha\in(0,1]}$ is unbounded. Then, there exits a subsequence $\{(z_{\alpha_n}, u_{\alpha_n})\}_{\alpha\in(0,1]} \subset \{(z_\alpha, u_\alpha)\}_{\alpha\in(0,1]}$ such that

$$\lim_{n\to\infty} |||(z_{\alpha_n}, u_{\alpha_n})||| = \infty.$$

On the other hand, from (4.28) we know for all $\alpha \in (0,1]$ that

$$\lim_{n\to\infty} \frac{|||\mathcal{K}^\alpha(z_{\alpha_n}, u_{\alpha_n})|||}{|||(z_{\alpha_n}, u_{\alpha_n})|||} = 0.$$

Particularly, we have the following situation:

$$\begin{array}{cccccc}
\frac{|||\mathcal{K}^{\alpha_1}(z_{\alpha_1},u_{\alpha_1})|||}{|||(z_{\alpha_1},u_{\alpha_1})|||} & \frac{|||\mathcal{K}^{\alpha_1}(z_{\alpha_2},u_{\alpha_2})|||}{|||(z_{\alpha_2},u_{\alpha_2})|||} & \frac{|||\mathcal{K}^{\alpha_1}(z_{\alpha_3},u_{\alpha_3})|||}{|||(z_{\alpha_3},u_{\alpha_3})|||} & \cdots\cdots & \frac{|||\mathcal{K}^{\alpha_1}(z_{\alpha_n},u_{\alpha_n})|||}{|||(z_{\alpha_n},u_{\alpha_n})|||} & \to 0. \\
\frac{|||\mathcal{K}^{\alpha_2}(z_{\alpha_1},u_{\alpha_1})|||}{|||(z_{\alpha_1},u_{\alpha_1})|||} & \frac{|||\mathcal{K}^{\alpha_2}(z_{\alpha_2},u_{\alpha_2})|||}{|||(z_{\alpha_2},u_{\alpha_2})|||} & \frac{|||\mathcal{K}^{\alpha_2}(z_{\alpha_3},u_{\alpha_3})|||}{|||(z_{\alpha_3},u_{\alpha_3})|||} & \cdots\cdots & \frac{|||\mathcal{K}^{\alpha_2}(z_{\alpha_n},u_{\alpha_n})|||}{|||(z_{\alpha_n},u_{\alpha_n})|||} & \to 0. \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\frac{|||\mathcal{K}^{\alpha_k}(z_{\alpha_1},u_{\alpha_1})|||}{|||(z_{\alpha_1},u_{\alpha_1})|||} & \frac{|||\mathcal{K}^{\alpha_k}(z_{\alpha_2},u_{\alpha_2})|||}{|||(z_{\alpha_2},u_{\alpha_2})|||} & \frac{|||\mathcal{K}^{\alpha_k}(z_{\alpha_3},u_{\alpha_3})|||}{|||(z_{\alpha_3},u_{\alpha_3})|||} & \cdots\cdots & \frac{|||\mathcal{K}^{\alpha_k}(z_{\alpha_n},u_{\alpha_n})|||}{|||(z_{\alpha_n},u_{\alpha_n})|||} & \to 0.
\end{array}$$

Now, applying Cantor's diagonalization process, we obtain that

$$\lim_{n\to\infty} \frac{|||\mathcal{K}^{\alpha_n}(z_{\alpha_n}, u_{\alpha_n})|||}{|||(z_{\alpha_n}, u_{\alpha_n})|||} = 0,$$

and from (4.29) we have that

$$\frac{|||\mathcal{K}^{\alpha_n}(z_{\alpha_n}, u_{\alpha_n})|||}{|||(z_{\alpha_n}, u_{\alpha_n})|||} = 1,$$

which is evidently a contradiction. Then, the claim is true and there exists $\gamma > 0$ such that

$$|||(z_{\alpha_n}, u_{\alpha_n})||| \leq \gamma, \quad (0 < \alpha \leq 1).$$

Therefore, without loss of generality, we can assume that the sequence $\mathcal{L}(z_\alpha, u_\alpha)$ converges to $y \in Z$. So, if

$$u_\alpha = \Gamma_\alpha \mathcal{L}(z_\alpha, u_\alpha) = G^*(\alpha I + GG^*)^{-1}\mathcal{L}(z_\alpha, u_\alpha).$$

Then,

$$\begin{aligned} Gu_\alpha &= G\Gamma_\alpha \mathcal{L}(z_\alpha, u_\alpha) = GG^*(\alpha I + GG^*)^{-1}\mathcal{L}(z_\alpha, u_\alpha) \\ &= (\alpha I + GG^* - \alpha I)(\alpha I + GG^*)^{-1}\mathcal{L}(z_\alpha, u_\alpha) \\ &= \mathcal{L}(z_\alpha, u_\alpha) - \alpha(\alpha I + GG^*)^{-1}\mathcal{L}(z_\alpha, u_\alpha). \end{aligned}$$

Hence,

$$Gu_\alpha - \mathcal{L}(z_\alpha, u_\alpha) = -\alpha(\alpha I + GG^*)^{-1}\mathcal{L}(z_\alpha, u_\alpha).$$

To conclude the proof of this Theorem, it enough to prove that

$$\lim_{\alpha \to 0}\{-\alpha(\alpha I + GG^*)^{-1}\mathcal{L}(z_\alpha, u_\alpha) = 0.$$

From Theorem 3.1.d we get that

$$\begin{aligned} \lim_{\alpha \to 0}\{\alpha(\alpha I + GG^*)^{-1}\mathcal{L}(z_\alpha, u_\alpha)\} &= \lim_{\alpha \to 0} \alpha(\alpha I + GG^*)^{-1}y \\ &+ \lim_{\alpha \to 0} \alpha(\alpha I + GG^*)^{-1}(\mathcal{L}(z_\alpha, u_\alpha) - y) \\ &= \lim_{\alpha \to 0} -\alpha(\alpha I + GG^*)^{-1}(\mathcal{L}(z_\alpha, u_\alpha) - y). \end{aligned}$$

On the other hand, from Proposition 3.1, we get that

$$\|\alpha(\alpha I + GG^*)^{-1}(\mathcal{L}(z_\alpha, u_\alpha) - y)\| \leq \|\mathcal{L}(z_\alpha, u_\alpha) - y)\|.$$

Therefore, since $\mathcal{L}(z_\alpha, u_\alpha)$ converges to $y$, we get that

$$\lim_{\alpha \to 0}\{-\alpha(\alpha I + GG^*)^{-1}(\mathcal{L}(z_\alpha, u_\alpha) - y)\} = 0.$$

Consequently,

$$\lim_{\alpha \to 0}\{-\alpha(\alpha I + GG^*)^{-1}\mathcal{L}(z_\alpha, u_\alpha)\} = 0.$$

Then,

$$\lim_{\alpha \to 0}\{Gu_\alpha - \mathcal{L}(z_\alpha, u_\alpha)\} = 0.$$

Therefore,

$$\lim_{\alpha\to 0}\{T(\tau)z_0+\int_0^\tau T(\tau-s)Bu_\alpha(s)ds \;+\; \int_0^\tau T(\tau-s)F(s,z_\alpha(s),u_\alpha(s))ds + \sum_{0<t_k<\tau} T(\tau-t_k)I_k^e(z_\alpha(t_k),u_\alpha(t_k))\} = z_1,$$

and the proof of the theorem is completed.

□

THEOREM 4.2. *The Impulsive Semilinear System (1.1) is approximately controllable if for all states $z_0$ and a final state $z_1$ and $\alpha \in (0,1]$ the operator $\mathcal{K}^\alpha$ given by (4.28)-(4.28) has a fixed point and the sequence $\{\mathcal{L}(z_\alpha,u_\alpha)\}_{\alpha\in(0,1]}$ converges. i.e.,*

$$(z_\alpha,u_\alpha)=\mathcal{K}^\alpha(z_\alpha,u_\alpha),$$

$$\lim_{\alpha\to 0}\mathcal{L}(z_\alpha,u_\alpha)=y\in Z.$$

## 5. Final Remark

Our technique is simple and can be apply to those system involving compact semigroups like some control system governed by diffusion processes. For example, the Benjamin -Bona-Mohany Equation, the strongly damped wave equations, beam equations, etc.

EXAMPLE 5.1. *The original Benjamin -Bona-Mohany Equation is a non-linear one, in [17] the authors proved the approximate controllability of the linear part of this equation, which is the fundamental base for the study of the controllability of the non linear BBM equation. So, our next work is concerned with the controllability of non linear BBM equation*

$$\begin{cases} z_t - a\Delta z_t - b\Delta z = u(t,x)+f(t,z,u(t)), & t\in(0,\tau), \quad x\in\Omega,\\ z(t,x)=0, \quad t\geq 0, & x\in\partial\Omega,\\ z(0,x)=z_0(x), x\in\Omega, \\ z(t_k^+,x)=z(t_k^-,x)+I_k(t,z(t_k,x),u(t_k,x)), x\in\Omega, \end{cases} \tag{5.30}$$

*where $a\geq 0$ and $b>0$ are constants, $k=1,2,\ldots,p$, $\Omega$ is a bounded domain in $\mathbb{R}^N(N\geq 1)$, $z_0\in L_2(\Omega)$, the distributed control $u$ belong to $C([0,\tau];L_2(\Omega;))$ and $f, I_k\in C([0,\tau]\times\mathbb{R}\times\mathbb{R};\mathbb{R})$, $k=1,2,3,\ldots,p$.*

## References


[1] J. BANAS AND K. GOEBEL, *Measures of Noncompactness in Banach Spaces.* Lecture Notes in Pure and Applied Mathematics, 60. Marcel Dekker, Inc., New York, 1980.

[2] D.D. BAINOV, V. LAKSHMIKANTHAM AND P.S. SIMEONOV, *Theory of Impulsive Differential Equations.* Simgapore: World Scienctific, 1989.

[3] D.BARCENAS, H. LEIVA AND Z. SIVOLI, *A Broad Class of Evolution Equations are Approximately Controllable, but Never Exactly Controllable.* IMA J. Math. Control Inform. 22, no. 3 (2005), 310–320.

[4] A. Carrasco, H. Leiva and J. Sanchez, *Controllability of the Semilinear Beam Equation*, Journal of Dynamical and Control Systems, Vol. 19, No. 3, October 2013.

[5] D. N. Chalishajar, *Controllability of Impulsive Partial Neutral Funcional Differential Equation with Infinite Delay.* Int. Journal of Math. Analysis, Vol. 5, 2011, N. 8, 369-380.

[6] LIZHEN CHEN AND GANG LI, *Approximate Controllability of Impulsive Differential Equations with Nonlocal Conditions.* International Journal of Nonlinear Science, Vol.10(2010), N. 4, pp. 438-446.

[7] R.F. CURTAIN, A.J. PRITCHARD, *Infinite Dimensional Linear Systems.* Lecture Notes in Control and Information Sciences, 8. Springer Verlag, Berlin (1978).

[8] R.F. CURTAIN, H.J. ZWART, *An Introduction to Infinite Dimensional Linear Systems Theory.* Text in Applied Mathematics, 21. Springer Verlag, New York (1995).

[9] LAWRENCE C. EVANS, *Partial Differential Equations.* Graduate Studies in Mathematics, Vol. 19, AMS. 1999.

[10] G. ISAC, "On Rothe's Fixed Point Theorem in General Topological Vector Space", An. St. Univ. Ovidius Constanta, Vol. 12(2), 2004, 127-134.

[11] C. KESAVAN, *Topics in: Functional Analysis and Applications.* John Wiley and Sons, 1989.

[12] H. LEIVA, "A Lemma on $C_0$-Semigroups and Applications PDEs Systems" Quaestions Mathematicae, Vol. 26, pp. 247-265 (2003).

[13] H. LEIVA"Controllability of a System of Parabolic equation with non-diagonal diffusion matrix". IMA Journal of Mathematical Control and Information; Vol. 32, 2005, pp. 187-199.

[14] H. LEIVA, N. MERENTES AND J.L. SANCHEZ,"Interior Controllability of the *nD* Semilinear Heat Equation". African Diaspora Journal of Mathematics, Special Vol. in Honor of Profs. C. Corduneanu, A. Fink, and S. Zaidman. Vol. 12, N. 2, pp. 1-12(2011).

[15] H. LEIVA, N. MERENTES and J. SANCHEZ "Approximate Controllability of Semilinear Reaction Diffusion" MATHEMATICAL CONTROL AND RALATED FIELDS, Vol. 2,N.2, June 2012.

[16] H. LEIVA, N. MERENTES and J. SANCHEZ "A Characterization of Semilinear Dense Range Operators and Applications", Abstract and Applied Analysis, Vol. 2013, Article ID 729093, 11 pages.

[17] H. LEIVA, N. MERENTES AND J.L. SANCHEZ "Interior Controllability of the Benjamin-Bona-Mahony Equation". Journal of Mathematis and Applications, N 33,pp. 51-59 (2010).

[18] A. PAZY *Semigroups of Linear Operators and Applications to Partial Differential Equations.*, Springer-Verlag, New York, (1983).

[19] N.A. PERESTYUK AND A.M. SAMOILENKO, *Impulsive Differential Equations.* Simgapore: World Scienctific, 1995.

[20] M.H. PROTTER, *Unique continuation for elliptic equations.* Transaction of the American Mathematical Society, Vol. 95, N 1, Apr., 1960.

[21] Bheeman RADHAKRISHNAN AND Krishnan BALACHANDRAN, *Controllability Results for Semilinear Impulsive Integrodifferential Evolution Systems with Nonlocal Conditions.*, J. Control Theory Appl. 2012, 10(1), 28-34.

[22] S. SELVI AND M. MALLIKA ARJUNAN, *Controllability Results for Impulsive Differential Systems with Finite Delay* J. Nonlinear Sci. Appl. 5 (2012), 206-219.

[23] J. D.R. SMART, *Fixed Point Theorems.* Cambridge University Press (1974).