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Regularized Evolution Operators and Applications

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Abstract: In this paper we introduce the notion of and give some results on regularized evolution operators, extending the definition of regularized quasisemigroups of bounded linear operators. Some applications of this theory are also discussed.

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1. INTRODUCTION

The theory of quasisemigroups of bounded linear operators was introduced in 1991 [1] as a generalization of strongly continuous semigroups of bounded linear operators. This notion with its elementary properties and some applications in abstract evolution equations are studied in [2, 6, 7, 8]. The dual quasisemigroups and the controllability of evolution equations are also discussed in [3].

Given a Banach space Z, we denote with L(Z) the space of all bounded linear operators from Z to Z. A bi-parametric family of bounded linear operators $\{K(t, s)\}_{t, s \geq 0} \subset L(Z)$ is called commutative if it satisfies:

$$K(r, t + s) = K(r + t, s)\, K(r, t) = K(r, t)\, K(r + t, s).$$

A commutative family $\{K(t, s)\}_{t,s \geq 0}$ is called strongly continuous quasisemigroups if it satisfies:

(i) $K(t, 0) = I$ $(t \geq 0)$ (I is the identity operator in L(Z)).

(ii) $K(r, t + s) = K(r + t, s)\, K(r, t)$ $(r, s, t \geq 0)$.

(iii) $\lim_{(t,s) \to (t_0,s_0)} \| K(t,s)z_0 - K(t_0,s_0)z_0 \| = 0$ $(z_0 \in Z)$.

(iv) There exists a continuous and increasing function $M : [0, \infty) \to [1, \infty)$ such that $\|K(t, s)\| \leq M(t + s)$ for every.

We also introduce the notion of a generator A(t) of strongly continuous quasisemigroups. These two notions generalize those of strongly continuous semigroups and infinitesimal generators. See [2].

Regularized semigroups and their connection with abstract Cauchy problems are introduced in [10], and have been studied in several articles, including [11, 15, 16, 17, 18]. This notion states the following: If $C \in L(Z)$ is an injective operator, then a one-parameter family of bounded linear operators $\{T(t)\}_{t \geq 0} \subseteq L(Z)$ is called a C-semigroup if it satisfies the following properties:

1. T(t) is strongly continuous, i.e, for each fixed $x \in Z$, $t \to T(t)x$ is continuous.
2. $T(t + s)C = T(t)T(s),\ \forall s,t \geq 0$
3. $T(0) = C$.

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The linear operators A defined by

$$D(A) = \left\{ x \in Z : C^{-1} \lim_{t \to 0^+} \frac{T(t) - Cx}{t} \text{exists and lies in } R(C) \right\}$$

and
$$A(t) = C^{-1} \lim_{t \to 0^+} \frac{T(t)x - Cx}{t} \quad \text{for } x \in D(A)$$

is the infinitesimal generator of the C-semigroup T(t) where D(A) is the domin of A.

The notion of regularized quasisemigroups of linear operators is introduced in [12], as an extension of the definition of C-Semigroup given above. Such notion is given as follows: Suppose C is an injective bounded linear operator on Banach space Z. A commutative bi-parametric family of bounded linear operators $\{K(t, s)\}_{s,t \geq 0} \subset L(Z)$ is called a regularized quasisemigroups (or C-quasisemigroups) if it satisfises,

(i) $K(t, 0) = C$ $(t \geq 0)$.

(ii) $CK(r, t + s) = K(r + t, s) K(r, t)$ $(r, s, t = 0)$.

(iii) $\lim_{(t,s) \to (t_0,s_0)} \| K(t,s)z_0 - K(t_0,s_0)z_0 \| = 0$ $(z_0 \in Z)$ That is, $\{K(t, s)\}_{t, s \geq 0}$, is strongly continuous.

(iv) There exists a continuous and increasing function $M : [0, \infty) \to [1, \infty)$ such that $\|K(t, s)\| \leq M(t + s)$ for every $s, t \geq 0$.

Let K(t, s) be a C-quasisemigroup and let D be the dense subspace of Banach space Z, such that for all $z_0 \in$ D there exist the limits in the range of C:

$$\lim_{s \to 0^+} \frac{K(t,s)z_0 - Cz_0}{s} = \lim_{s \to 0^+} \frac{K(t,-s,s)z_0 - Cz_0}{s}, t > 0, \lim_{s \to 0} \frac{K(0,s)z_0 - Cz_0}{s}$$

The family of operators A(t), t = 0, with common domain D, defined by

$$A(t)x_0 = C^{-1} \lim_{s \to 0^+} \frac{K(t,s)z_0 - Cz_0}{s} \quad (z_0 \in D),$$

is called the generator of the regularized quasisemigroup K(t, s).

In order to motivate our work, we observe that given a quasisemigroup, one may (according to definition 2.3) associate to it an evolution operator defined by

$$U(t, s) = K(t, t - s),$$

but not the other way around since in general an evolution operator U(t, s) is not a commutative family. This observation and the foregoing theory of regularized quasisemigroups allows as to introduce and study the notion of regularized evolution operators, and we generalize Lemma 3.1 from [5].

Finally, as an application, we will be interested in the abstract Cauchy problem defined on a Banach space Z,

$$\begin{cases} z'(t) = A(t)z(t) + C^2 f(t), 0 \leq s \leq t < \infty \\ z(s) = C^2 z_0 \in Z, \end{cases} \tag{1.1}$$

where $z : [0, \infty) \to Z$, A(t) is a family of unbounded linear operators in Z with domain D(A(t)) = D, independent of t, such that $A(\cdot)z \in C(\mathbb{R}^+, Z)$ for each $z \in$ D, $f : [0, T] \to Z$ is suitable function and C is an injective linear operator in Z.

As an example of this problem we study the following system of reaction diffusion equations:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = D\Delta(t,x) + B(t)u(t,x) + C^2 f(t,x), t>0, u \in \mathbb{R}^n, \\ \dfrac{\partial u(t,x)}{\partial \eta} = 0, t>0, x, \in \ \partial\Omega \\ u(0,x) = C^2\psi(x), x \in \Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$, $\alpha, \psi \in L^2(\in)$, with $\alpha(x) > 0$, $\forall x \in \Omega$ and D is an n × n matrix whose eigenvalues are semisimple with non negative real part or strictly positive and $f : \mathbb{R} \times \Omega \to \mathbb{R}^n$ is a smooth function. We assume that the operator $B \in P_\infty[0,\infty);\Gamma(Z))$:

$$\left\{ B / \langle z_1, B(\cdot)z_2 \rangle \text{ is measurable} \forall z_1, z_2 \in Z \text{ and } (\operatorname*{ess\,sup}_{0 \leq t < \infty}) \| B(t) \|_{\Gamma(z)} < \infty \right\}$$

with $Z = L^2(\Omega)$ and $C : L^2(\Omega) \to L^2(\Omega)$ is the linear operator given by $(C\psi)(x) = \alpha(x)\psi(x)$.

2. EVOLUTION OPERATORS

In this section we are interested in the classical abstract Cauchy problem defined on a Banach space Z,

$$\begin{cases} z'(t) = A(t)z(t), 0 \leq s \leq t < \infty \\ z(s) = z_0 \in Z, \end{cases} \tag{2.1}$$

where $z : [0, \infty) \to Z$, $A(t)$ is a family of unbounded linear operators in Z with domain $D(A(t)) = D$, independent of t, such that $A(\cdot)z \in (\mathbb{R}^+, Z)$ for each $z \in D$.

Definition 2.1: (see [14]) An operator-valued function $U(t, s) \in \mathcal{L}(Z)$ which is strongly continuous jointly in t, s for $0 \leq s \leq t < \infty$, is called fundamental solution of (2.1) if

1. For all $z \in D$ the partial derivative $\dfrac{\partial U(t,s)z}{\partial t}$ exists in the strong topology of Z and it is strongly continuous in (t, s) for $0 \leq s \leq t < \infty$.
2. For all $z \in D$, $U(t, s)z \in D$.
3. For all $z \in D$, $= \dfrac{\partial U(t,s)z}{\partial t}$ $A(t)\,U(t, s)z$, $0 \leq s \leq t < \infty$ and $U(s, s) = I$.

Proposition 2.2: The operator-valued function U(t, s) given by the foregoing definition satisfies the following properties:

$$U(t,s)z_0 = z_0 + \int_s^t A_{(\tau)} U_{(\tau,s)z_0 d\tau}, \ \forall z_0 \in D, \tag{2.2}$$

$$\lim_{r\to t^+}\lim_{k\to t^-} \frac{U(r,k)U(k,s)_{z_0} - U(k,s)_{z_0}}{r-k} = \lim_{k\to t^-}\lim_{r\to t^+} \frac{U(r,k)U(k,s)_{z_0} - U(k,s)_{z_0}}{r-k} \tag{2.3}$$

The above calculation motivates the following definitions.

Definition 2.3: A two-parameter family of bounded linear operators $U(t, s) \in \mathcal{L}(Z)$, $0 \leq s \leq t < \infty$ see [9] is called an evolution operator if the following conditions are satisfied:

1. $U(s, s) = I$ and $U(t, r)\,U(r, s) = U(t, s)$ for $0 \leq s \leq t < \infty$.

2. $(t, s) \to U(t, s)$ is strongly continuous for $0 \le s \le t < \infty$.
3. There exists a real valued continuous nonnegative function $g(t, s)$ with $\|U(t, s)\| \le g(t, s)$ for all $0 \le s \le t < \infty$.

Proposition 2.4: Given a quasisemigroups $K(t, s)$ the evolution operator defined by

$$U(t, s)z_0 = K(t, t - s)z_0, \quad \forall z_0 \in D, t \ge s$$

satisfies:

$$\lim_{r\to t^+}\lim_{k\to t^-}\frac{U(r,k)U(k,s)_{z_0} - U(k,s)_{z_0}}{r-k} = \lim_{k\to t^-}\lim_{r\to t^+}\frac{U(r,k)U(k,s)_{z_0} - U(k,s)_{z_0}}{r-k}$$

Proof: In fact,

$$\lim_{r\to t^+}\lim_{k\to t^-}\frac{U(r,k)U(k,s)_{z_0} - U(k,s)_{z_0}}{r-k} = \lim_{r\to t^+}\lim_{k\to t^-}\frac{K(r,r-k)K(k,k-s)_{z_0} - K(k,k-s)_{z_0}}{r-k}$$

$$= \lim_{r\to t^+}\lim_{k\to t^-}K(t,t-s)\frac{(K(r,r-k)z_0 - z_0)}{r-k} = \lim_{r\to t^+}K(t,t-s)\frac{(K(r,r-t)z_0 - z_0)}{r-t}$$

$$= \lim_{h\to 0^+}K(t,t-s)\frac{(K(t+h,h)z_0 - z_0)}{h} = K(t, t - s)\, A(t)z_0 = U(t, s)\, A(t)z_0,$$

and

$$\lim_{k\to t^-}\lim_{r\to t^+}\frac{U(r,k)\,U(k,s)z_0 - U(k,s)z_0}{r-k} = \lim_{k\to t^-}\lim_{r\to t^+}\frac{K(r,r-k)K(k,k-s)z_0 - K(k,k-s)z_0}{r-k}$$

$$= \lim_{k\to t^-}\lim_{r\to t^+}K(k,k-s)\frac{(K(r,r-k)z_0 - z_0)}{r-k} = \lim_{k\to t^-}K(k,k-s)\frac{(K(t,t-k)z_0 - z_0)}{t-k}$$

$$= \lim_{h\to 0^+}K(t-h,t-h-s)\frac{(K(t,h)z_0 - z_0)}{h} = K(t, t - s)\, A(t) = U(t, s)\, A(t)\, z_0.$$

Motivated by Propositions 2.2 and 2.4 we define the following important subspace:

Definition 2.5: We define the subspace D consisting of all $z \in Z$ such that the following limits

$$\lim_{r\to t^+}\lim_{k\to t^-}\frac{U(r,k)U(k,s)z - U(k,s)z}{r-k} = \lim_{k\to t^-}\lim_{r\to t^+}\frac{U(r,k)U(k,s)z - U(k,s)z}{r-k}$$

exist for all $0 \le s \le t < \infty$.

Remark 2.6: For all $z \in D$ and $t \ge 0$ the following limits exists

$$\lim_{h\to 0^+}\frac{U(t+h,t)z - z}{h} = \lim_{r\to t^+}\frac{U(r,t)z - z}{r-t}$$

In fact, since z belongs to D we have the existence of the limits

$$\lim_{r\to t^+}\lim_{k\to t^-}\frac{U(r,k)U(k,s)z - U(k,s)z}{r-k} = \lim_{k\to t^-}\lim_{r\to t^+}\frac{U(r,k)U(k,s)z - U(k,s)z}{r-k}$$

for all $0 \le s \le t < \infty$. Then if we put $s = t$ and make a change of variable, we get

$$\lim_{h\to 0^+}\frac{U(t+h,t)z - z}{h} = \lim_{r\to t^+}\frac{U(r,t)z - z}{r-t}$$

Now, we shall give a definition of generator of an evolution operator, which is similar to the one given in [13] pg 1902.

Definition 2.7: The generator A(t) of an evolution operator U(t, s), $0 \le s \le t < \infty$ is defined as follows:

$$A(t)z = \lim_{h \to 0^+} \frac{U(t+h,t)z - z}{h}, \ \forall z \in D, \ 0 \le t < \infty.$$

Thus, we get that D(A(t)) = D.

3. REGULARIZED EVOLUTION OPERATORS

In this section we are interested in the generalized abstract Cauchy problem defined on a Banach space Z,

$$\begin{cases} z'(t) = A(t)z(t), 0 \le s \le t < \infty \\ z(s) = C^2 z_0 \in Z, \end{cases} \tag{3.1}$$

where $z : [0, \infty) \to Z$, A(t) is a family of unbounded linear operators in Z with domain D(A(t)) = D, independent of t, such that $A(\cdot)z \in C(\mathbb{R}^+, Z)$ for each $z \in D$ and C is a bounded linear operator in Z.

Definition 3.1: Suppose C is an injective bounded linear operator on the Banach space Z. A two-parameter family of bounded linear operators $U(t, s) \in \mathcal{L}(Z)$, $0 \le s \le t < \infty$, is called regularized evolution operator (or C-Evolution Operator) if the following conditions are satisfied:

1. U(s, s) = C and CU(t, s) = U(t, r) U(r, s) for $0 \le s \le t < \infty$.
2. $(t, s) \to U(t, s)$ is strongly continuous for $0 \le s \le t < \infty$.
3. There exists a real valued continuous nonnegative function g(t, s) with $\|U(t, s) \le g(t, s)$ for all $0 \le s \le t < \infty$.

Definition 3.2: We define the subspace D consisting of all $z \in Z$ such that the following limits

$$\lim_{r \to t^+} \lim_{k \to t^-} \frac{U(r,k)U(k,s)z - CU(k,s)z}{r-k} = \lim_{k \to t^-} \lim_{r \to t^+} \frac{U(r,k)U(k,s)z - CU(k,s)z}{r-k}$$

exist for all $0 \le s \le t < \infty$.

Remark 3.3: For all $z \in D$ and we have the existence of this limits

$$\lim_{h \to 0^+} \frac{U(t+h,t)z - Cz}{h} = \lim_{h \to t^+} \frac{U(r,t)z - Cz}{r-t}$$

In fact, since z belongs to D we have the existence of the limits

$$\lim_{k \to t^+} \lim_{k \to t^-} \frac{U(r,k)U(k,s)z - CU(k,s)z}{r-k} = \lim_{k \to t^-} \lim_{k \to t^+} \frac{U(r,k)U(k,s)z - CU(k,s)z}{r-k}$$

Then, if we put s = t and make a change of variable, we get

$$\lim_{h \to 0^+} \frac{U(t+h,t)z - Cz}{h} = \lim_{r \to t^+} \frac{U(r,t)z - Cz}{r-t}$$

Definition 3.4: The generator A(t) of a regularized evolution operator U(t, s), $0 \le s \le t < \infty$ is defined as follows:

$$A(t)z = C^{-1} \lim_{h \to 0^+} \frac{U(t+h,t)z - Cz}{h}, \ \forall z \in D, 0 \le t < \infty.$$

So, we get that D(A(t)) = D.

Lemma 3.5: Let U(t, s), $0 \leq s \leq t < \infty$ be a regularized evolution operator on Z and C is an injective bounded linear operator on Banach space Z, such that, U(t, s) Cz,C^{-1} U(t, s) $z \in D$ for all $z \in D$, furthermore CU(t, s) = U(t, s)C and C^{-1} U(t, s) = U(t, s) C^{-1}. Then for all $z \in D$ we have that

$$\frac{\partial}{\partial t}U(t,s)Cz = A(t)U(t,s)Cz$$

and
$$\frac{\partial}{\partial s}U(t,s)Cz = -A(s)U(t,s)Cz.$$

$$\frac{\partial}{\partial t}C^{-1}U(t,s)z = C^{-1}A(t)U(t,s)z$$

and
$$\frac{\partial}{\partial s}C^{-1}U(t,s)z = -C^{-1}A(s)U(t,s)z$$

Proof: If $z \in D$, the from the hypothesis we have that U(t, s)Cz, C^{-1} U(t, s) $z \in D$,

$$\lim_{h\to 0^+}\frac{U(t+h,s)Cz-U(t,s)Cz}{h} = \lim_{h\to 0^+}\frac{C^{-1}CU(t+h,s)Cz-U(t,s)Cz}{h}$$

$$= C^{-1}\lim_{h\to 0^+}\frac{u(t+h,t)U(t,s)Cz-CU(t,s)Cz}{h}$$

Since U(t, s) $Cz \in D$, we obtain that

$$\lim_{h\to 0^+}\frac{U(t+h,t)U(t,s)Cz-U(t,s)Cz}{h} = A(t)U(t,s)Cz.$$

Now, suppose $t > s$ and $h \geq 0$ is small enough such that $t - h \geq s$. Then

$$\lim_{h\to 0^+}\frac{U(t-h,s)Cz-U(t,s)Cz}{-h} = \lim_{h\to 0^+}\frac{U(t,s)Cz-U(t-h,s)Cz}{h}$$

$$= \lim_{h\to 0^+}\frac{C^{-1}CU(t,s)Cz-U(t-h,s)Cz}{h}$$

$$= \lim_{h\to 0^+}\frac{C^{-1}U(t,t-h)U(t-h,s)Cz-U(t-h,s)Cz}{h}$$

$$= \lim_{k\to t^-}\frac{C^{-1}U(t,k)U(k,s)Cz-U(k,s)Cz}{t-k}$$

$$= \lim_{r\to t^+}\lim_{k\to t^-}\frac{C^{-1}U(r,k)U(k,s)Cz-U(k,s)Cz}{r-k}$$

$$= \lim_{r\to t^+}\frac{C^{-1}U(r,t)U(t,s)Cz-U(t,s)Cz}{r-t}$$

$$= C^{-1}\lim_{h\to 0^+}\frac{U(t+h,t)U(t,s)Cz-CU(t,s)Cz}{h}$$

$$= A(t)U(t,s)Cz.$$

So, $$\frac{\partial}{\partial t}U(t,s)Cz = A(t)\,U(t,s)\,Cz \text{ for } 0 \le s < t < \infty.$$

Again, suppose that $t > s$ and $h \ge 0$ is small enough such that $s + h < t$. Then

$$\left\|\frac{U(t,s+h)Cz - U(t,s)Cz}{h} + U(t,s)A(s)Cz\right\| = \left\|-\left[\frac{C^{-1}CU(t,s)Cz - U(t,s+h)Cz}{h} - C^{-1}CU(t,s)A(s)Cz\right]\right\|$$

$$= \left\|-\left[\frac{C^{-1}U(t,s+h)U(s+h,s)Cz - U(t,s+h)Cz}{h} - C^{-1}U(t,s+h)U(s+h,s)A(s)Cz\right]\right\|$$

$$= \left\|-U(t,s+h)\left[C^{-1}\frac{U(s+h,s)Cz - C(Cz)}{h} - C^{-1}U(s+h,s)A(s)Cz\right]\right\|$$

$$\le g(t,s+h)\left\|\left[C^{-1}\cdot\frac{U(s+h,s)Cz - C(Cz)}{h} - C^{-1}U(s+h,s)A(s)Cz\right]\right\|$$

Since $C^{-1}\lim\limits_{h\to 0^+}\dfrac{U(s+h,s)Cz - C(Cz)}{h} = A(S)\,Cz$, and

$$C^{-1}\lim_{h\to 0^+}U(s+h,s)A(s)Cz = C^{-1}CA(s)Cz = A(s)Cz,$$

we obtain that

$$\lim_{h\to 0^+}\frac{U(t,s+h)Cz - U(t,s)Cz}{h} = -U(t,s)\,A(s)\,Cz.$$

Similarly,

$$\frac{U(t,s-h)Cz - U(t,s)Cz}{-h} = \frac{U(t,s)Cz - U(t,s-h)Cz}{h}$$

$$= \frac{U(t,s)Cz - C^{-1}CU(t,s-h)Cz}{h}$$

$$= \frac{U(t,s)Cz - C^{-1}U(t,s)U(s,s-h)Cz}{h}$$

$$= -C^{-1}\cdot\frac{U(t,s)U(s,s-h)\,Cz - CU(t,s)Cz}{h}$$

$$= -C^{-1}\cdot U(t,s)\left[\frac{U(s,s-h)Cz - C(Cz)}{h}\right]$$

We also have

$$\lim_{h\to 0^+}\frac{U(t,s-h)Cz - U(t,s)Cz}{-h} = -U(t,s)\cdot C^{-1}\left[\lim_{h\to 0^+}\frac{U(s,s-h)Cz - C(Cz)}{h}\right]$$

$$= -U(t,s)\,A(s)z.$$

so that $\dfrac{\partial U(t,s)Cz}{\partial s} = -U(t,s)A(s)Cz$ for $0 \le s < t < \infty$ Now,

$$\lim_{h\to 0^+}\frac{C^{-1}U(t+h,s)z-C^{-1}U(t,s)z}{h} = \lim_{h\to 0^+}\frac{C^{-1}C^{-1}CU(t+h,s)z-C^{-1}U(t,s)z}{h}$$

$$= C^{-1}\cdot\lim_{h\to 0^+}\frac{C^{-1}U(t+h,t)U(t,s)z-U(t,s)z}{h}$$

$$= C^{-1}\left[C^{-1}\cdot\lim_{x\to 0^+}\frac{U(t+h,t)U(t,s)z-CU(t,s)z}{h}\right]$$

$$= C^{-1}A(t)U(t,s)z.$$

Now, suppose $t > s$ and $h \geq 0$ is small enough such that $t - h \geq s$. Then

$$\lim_{h\to 0^+}\frac{C^{-1}U(t-h,s)z-C^{-1}U(t,s)z}{-h} = C^{-1}\cdot\lim_{h\to 0^+}\frac{C^{-1}CU(t,s)z-U(t-h,s)z}{h}$$

$$= C^{-1}\left[C^{-1}\cdot\lim_{h\to 0^+}\frac{U(t,t-h)U(t-h,s)z-CU(t-s,s)z}{h}\right]$$

$$= C^{-1}\left[C^{-1}\lim_{k\to t^-}\frac{U(t,k)U(k,s)z-CU(k,s)z}{r-k}\right]$$

$$= C^{-1}\left[C^{-1}\lim_{k\to t^-}\lim_{r\to t^+}\frac{U(r,k)U(k,s)z-CU(k,s)z}{r-k}\right]$$

$$= C^{-1}\left[C^{-1}\lim_{r\to t^+}\lim_{k\to t^-}\frac{U(r,k)U(k,s)z-CU(k,s)z}{r-k}\right]$$

$$= C^{-1}\left[C^{-1}\lim_{r\to t^+}\frac{U(r,t)U(t,s)z-CU(t,s)z}{r-t}\right]$$

$$= C^{-1}\left[C^{-1}\lim_{h\to 0^+}\frac{U(t+h,t)U(t,s)z-CU(t,s)z}{h}\right]$$

$$= C^{-1}A(t)U(t,s)z.$$

So,

$$\lim_{h\to 0^+}\frac{C^{-1}U(t-h,s)z-C^{-1}U(t,s)z}{-h} = C^{-1}A(t)U(t,s)z.$$

Therefore, $$\frac{\partial}{\partial t}C^{-1}U(t,s) = C^{-1}A(t)U(t,s)z.$$

Analogously

$$\frac{C^{-1}U(t,s-h)z-C^{-1}U(t,s)z}{-h} = C^{-1}\left[\frac{U(t,s)z-C^{-1}U(t,s)U(t,s-h)z}{h}\right]$$

$$= -C^{-1}\left[\frac{C^{-1}U(t,s)U(s,s-h)z-U(t,s)z}{h}\right]$$

$$= -C^{-1}\left[C^{-1}U(t,s)\left(\frac{U(s,s-h)z-Cz}{h}\right)\right].$$

We also have $C^{-1}\lim_{h\to 0^+}\left(\frac{U(s,s-h)z-Cz}{h}\right)=U(t,s)A(s)z$. And we obtain

$$\lim_{h\to 0^+}\frac{C^{-1}U(t,s-h)z-C^{-1}U(t,s)z}{-h}=-C^{-1}U(t,s)A(s)z.$$

Again, suppose that t > s and h ≥ 0 is small enough such that s + h < t. Then

$$\left\|\frac{C^{-1}U(t,s+h)z-C^{-1}U(t,s)z}{h}+C^{-1}U(t,s)A(s)z\right\|$$

$$=\left\|C^{-1}\left[\frac{U(t,s+h)z-C^{-1}U(t,s+h)U(s,s+h)z}{h}+C^{-1}U(t,s+h)\,U(s+h,s)A(s)Cz\right]\right\|$$

$$=\left\|-C^{-1}\{U(t,s+h)\left[\frac{C^{-1}U(s+h,s)z-z}{h}-C^{-1}U(s+h,s)A(s)z\right]\right\|$$

$$\le \|C^{-1}\|\,\|U(t,s+h)\|\left\|\frac{C^{-1}U(s,s+h)z-z}{h}-C^{-1}U(s+h,s)A(s)Cz\right\|$$

$$\le Mg(t,s+h)\left\|\frac{C^{-1}U(s,s+h)z-z}{h}-C^{-1}U(s+h,s)A(s)z\right\|$$

$$= Mg(t,s+h)\left\|C^{-1}\left[\frac{U(s,s+h)z-Cz}{h}-C^{-1}U(s+h,s)A(s)z\right]\right\|.$$

Since

$$C^{-1}\lim_{h\to 0^+}\frac{U(s,s+h)z-Cz}{h}=A(s)\text{ and }C^{-1}\lim_{h\to 0^+}U(s+h,s)A(s)z$$

we obtain

$$\lim_{h\to 0^+}\frac{C^{-1}U(t,s+h)z-C^{-1}U(t,s)z}{h}=-C^{-1}U(t,s)A(s)z.$$

Therefore $$\frac{\partial}{\partial s}C^{-1}U(t,s)z=-C^{-1}A(s)U(t,s)z.$$

Proposition 3.6: The C-evolution operator U(t, s) given by the foregoing definition satisfies the following properties:

$$U(t,s)z_0=Cz_0+\int_s^t A_{(\tau)}U_{(\tau,S)z_0 d\tau,}\ \forall z_0\in D. \qquad (3.2)$$

Theorem 3.7: Let U(t, s), 0 = s = t < ∞ be a regularized evolution operator on Z satisfying the condition Lemma 3.5 and C is an injective bounded linear operator on Banach space Z and let A(t) its generator with domain D. Then the Cauchy problem

$$\begin{cases} z'(t)=A(t)z(t)\ t\ge s,\\ z(s)=C^2z_0,\ z_0\in Z.\end{cases} \qquad (3.3)$$

has the unique solution the function

$$z(t) = U(t,\ s)Cz_0.$$

Proof: From Lemma 3.5 we get that $z(t) = U(t, s)Cz_0$ is one solution of the Cauchy problem (3.3). Now, to prove uniqueness, we will suppose that y(t) is another solution of the problem, then w(t) = z(t) – y(t) satisfies the differential equation

$$\frac{dw}{dt} = A(t)w(t),\ t \geq s;\ w(s) = 0$$

and so we need to show that $w(t) \equiv 0$. For this, let us define F(u) = U(t, u) Cw(u), $0 \leq u = s < t$. Then

$$\begin{aligned} F'(u) &= \frac{\partial}{\partial u} U(t,u)\, Cw(u) + U(t,u)C \frac{d}{du} w(u) \\ &= -A(u)U(t,u)\, C\, w(u) + U(t,u)C\, A(u)w(u) \\ &= 0. \end{aligned}$$

Therefore, F(u) = U(t, u) Cw(u) = c (constant). In particular, if we put u = s, we have that F(s) = U(t, s) Cw(s) = 0. Now, from the strongly continuity of U(t, s) we get that $F(t) = \lim_{s \to t^-} F(s) = \lim_{s \to t^-} U(t,s)\, Cw(s) = 0$. So,

$$F(t) = 0 \quad \Rightarrow U(t, t)\, Cw(t) = 0 \Rightarrow C^2 w(t) = 0$$

Now, injectivity of C implies that w(t) = 0, we conclude that

$$z(t) = w(t)$$

which proves the uniqueness of the solution

Proposition 3.8: Let U(t, s), $0 \leq s \leq t < \infty$ be regularized evolution operator on Z and $C \in L(Z)$ injective with A(t) its generator closed on D. If f : [0, T] $\to$ Z is continuous, then

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} U(u,s)Cf(u)du = U(t, s)\, Cf(t) \tag{3.4}$$

$$\lim_{h \to 0} \frac{1}{h} \int_s^{s+h} U(t,u)Cf(u)du = U(t, s)\, Cf(s). \tag{3.5}$$

Proof: We prove (3.4), the proof of (3.5) is similar. s, t $\geq$ 0. We define the following function

$$\varphi(t) = U(t, s)\, Cf(t).$$

φ is clearly continuous in s. Hences

$$F(\epsilon) = \int_t^{t+\epsilon} \varphi(u)du\ (\epsilon > 0$$

is well defined. Now,

$$F'(0) = \lim_{h \to 0} \frac{F(0+h) - F(0)}{h} = \lim_{h \to 0} \frac{1}{h} \int_t^{t+h} \varphi(u)du,\ (h > 0$$

and for each $\epsilon > 0$ we have

$$F'(\epsilon) = \frac{d}{d\epsilon} \int_t^{t+\epsilon} \varphi(u)du = \varphi(t+\epsilon) = U(t+\epsilon, s)Cf(t+\epsilon)$$

Putting $\epsilon > 0$,

$$F'(0) = U(t, s)\, Cf(t).$$

Therefore, $\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} \varphi(u)du = U(t, s)\, Cf(t).$

Proposition 3.9: Let $U(t, s)$, $0 \le s \le t < \infty$ be a regularized evolution operator commutative on Z and $C \in L(Z)$ injective with $A(t)$ its generator closed on D. If $f : [0, T] \to Z$ is continuous then function. If $\{x_n\}$ is a sequence in D such that $x_n \to x$ and $A(t)x_n$ converges uniformly to $f(t)$ in $[0, T]$, then for each we have that $A(r)z = f(r)$.

Proof: Let be $r \in [0, t]$, and $s > 0$. By definition,

$$A(r)x_n = C^{-1} \lim_{s \to 0^+} \frac{U(r+h, r)x_n - Cx_n}{s}.$$

Passing to the limit as $n \to \infty$ $A(r)x_n \to A(r)x$. Now, from the uniform convergence and the Proposition 3.6,

$$\begin{aligned} U(r + s, r)x - Cx &= \lim_{n \to \infty}(U(r+s, r)x_n - Cx_n) \\ &= \lim_{n \to \infty} \int_r^{r+s} A(\tau)U(\tau, r)x_n d\tau \\ &= \lim_{x \to \infty} \int_r^{r+s} U(\tau, r)A(\tau)x_n d\tau \\ &= \lim_{x \to \infty} \int_r^{r+s} U(\tau, r)A(\tau)x_n d\tau \\ &= \int_r^{r+s} U(\tau, r) f(\tau) d\tau \end{aligned}$$

Dividing by $s > 0$,

$$\frac{U(r+s, r)x - Cx}{s} = \frac{1}{s}\int_r^{r+s} U(\tau, r) f(\tau) d\tau,$$

Consider $C = I$ in the Proposition 3.8 and taking limits as $s \to 0^+$

$$\begin{aligned} A(r)x &= C^{-1} \lim_{s \to 0^+} \frac{U(r+s, r)x - Cx}{s} \\ &= C^{-1}\left(\lim_{s \to 0^+} \frac{1}{s}\int_r^{r+s} U(\tau, r) f(\tau)\, d\tau\right) \\ &= f(r). \end{aligned}$$

Theorem 3.10: Let $U(t, s)$, $0 \le s \le t < \infty$ be a regularized evolution operator on Z and $C \in L(Z)$ injective with $A(t)$ its generator closed on D. If $f : [0, T] \to D$ is a continuously differentiable function and

$$\int_s^t U(r, u)C\ f(u)\, du \in D.$$

Consider the non-homogeneous Cauchy problem,

$$\begin{cases} z'(t) = A(t)z(t) + C^2 f(t), t \ge s, \\ z(s) = C^2 z_0, z_0 \in Z, 0 \le s \le t. \end{cases} \tag{3.6}$$

Then (3.6) admits as unique solution the function

$$z(t) = U(t, s)Cz_0 + \int_s^t U(t, \alpha)C\ f(\alpha)d\alpha. \tag{3.7}$$

Proof: For the existence of the solution it is enough to prove that the function z(t) in (3.7) has strong derivative and satisfies the equation (3.6). Clearly $z(s) = C^2 z_0$. On the other hand, the function $y(t) = U(t, s)Cz_0$ is a solution of the initial value problem (3.3).

So we need prove that

$$\int_s^t U(t,\alpha)Cf(\alpha)d\alpha,$$

satisfies the initial value problem (3.6). Indeed, let us define the function

$$g(t) = \int_s^t U(t,\alpha)Cf(\alpha)d\alpha$$

Since U(t, s) is strongly continuous evolution operator and $f(\cdot)$ is continuous, the above integral exists. Let $h \neq 0$ and consider the following quotient

$$\frac{g(t+h)-g(t)}{h} = \frac{1}{h}\left(\int_s^{t+h} U(t+h,\alpha)Cf(\alpha)d\alpha - \int_s^t U(t,\alpha)Cf(\alpha)d\alpha\right)$$

$$= \frac{1}{h}\left(\int_s^t (U(t+h,\alpha)-U(t,\alpha)\, Cf(\alpha)d\alpha\right) + \frac{1}{h}\int_t^t U(t+h,\alpha)Cf(\alpha)d\alpha.$$

But

$$\lim_{h\to 0}\frac{1}{h}\left(\int_s^t (U(t+h,\alpha)-U(t,\alpha))Cf(\alpha)d\alpha\right)$$

$$= \int_s^t \lim_{h\to 0}\frac{(U(t+h,\alpha)-U(t,\alpha))}{h}Cf(\alpha)d\alpha = \int_s^t A(t)U(t,\alpha)Cf(\alpha)d\alpha;$$

and

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} U(t+h,\alpha)Cf(\alpha)d\alpha = C^2 f(t)$$

So, if $\int_s^t U(t,\alpha)Cf(\alpha)d\alpha \in D$, then

$$g'(t) = \int_s^t A(t)U(t,\alpha)Cf(\alpha)d\alpha + C^2 f(t)$$

$$= A(t)\int_s^t U(t,\alpha)Cf(\alpha)d\alpha + C^2 f(t)$$

$$= A(t)g(t) + C^2 f(t).$$

Therefore:

$$z'(t) = \frac{\partial}{\partial t}U(t,s)Cz_0 + A(t)\int_s^t U(t,\alpha)Cf(\alpha)d\alpha + C^2 f(t)$$

$$= A(t)U(t,s)Cz_0 + \int_s^t U(t,\alpha)Cf(\alpha)d\alpha + C^2 f(t)$$

$$= A(t)\left(U(t,s)Cz_0 + \int_s^t U(t,\alpha)Cf(\alpha)d\alpha\right) + C^2 f(t)$$

$$= A(t)\, z(t) + C^2 f(t).$$

The uniqueness is consequence of Theorem 3.7.

4. A LEMMA ON REGULARIZED EVOLUTION OPERATORS

In this section we shall characterize a family of regularized evolution operators that can be used to and a variation constants formula for a broad class of nonautonomous partial differential equations.

Lemma 4.1: Let Z be a Hilbert Space, $\{U_n(t, s)\}_{0 \le s \le t < \infty}$ a family of regularized evolution operators and $P_n(\cdot) : [0, \infty) \to L(Z)$; n = 1, 2, . . ., a family of strongly continuous orthogonal projections on Z, which are complete and

$$P_n(t)U_n(t, s) = U_n(t, s)\, P_n(s);\ n = 1, 2, \ldots, \ 0 \le s \le t < \infty.$$

Let us define the following family of linear operators

$$U(t, s) = \sum_{n=1}^{\infty} U_n(t,s)P_n(s)z,\ 0 \le s \le t < \infty.$$

Then, the following statements holds:

(i) $\{U(t, s)\}_{0 \le s \le < \infty}$ is a regularized evolution operator, if $\|U_n(t, s)\| \le g(t, s)$, n = 1, 2, . . ., with $g(t, s) \ge 0$, continuous in $0 \le s \le t < \infty$.

(ii) The generator $A(t) : D \to Z$ of $\{U(t, s)\}_{0 \le s \le t < \infty}$ is given by

$$A(t)z = \sum_{n=1}^{\infty} A_n(t)P_n(t)z,\ z \in D,$$

where

$$D \subset W = \left\{ z \in Z : \sum_{n=1}^{\infty} \| A_n(t)P_n(t)z \|^2 < \infty.\ \forall t \in [0, \infty \right\}.$$

and if A(t) is a closed operator, then D = W.

(iii) Suppose A(t) is a closed operator. If $z \in D$, then $U(t, s)\, z \in D$.

Proof: We show first that U(t, s) is a bounded linear operator for fixed s = t. In fact, let $z \in Z$. Then

$$\begin{aligned}
\|U(t, s)z\|^2 &= \left\langle \sum_{n=1}^{\infty} U_n(t,s)P_n(s)z, \sum_{m=1}^{\infty} U_m(t,s)P_m(s)z \right\rangle \\
&= \sum_{n,m=1}^{\infty} \langle U_n(t,s)P_n(s)z, U_m(t,s)P_m(s)z \rangle \\
&= \sum_{n,m=1}^{\infty} \langle P_n(t)U_n(t,s)P_n(s)z, P_m(t)U_m(t,s)P_m(s)z \rangle \\
&= \sum_{n=1}^{\infty} \| U_n(t,s)P_n(s)z \|^2 \\
&\le (g(t, s))^2\, \|z\|^2.
\end{aligned}$$

This proves that U(t, s) is bounded.

Now, we will show that U(t, r)U(r, s) = CU(t, s) for 0 = r = s = t < ∞

$$U(t, r)U(r, s)z = \sum_{n=1}^{\infty} U_n(t,r)P_n(r) \left(\sum_{i=1}^{\infty} U_i(r,s)P_i(s)z \right)$$

$$= \sum_{n=1}^{\infty} U_n(t,r)U_n(r,s)P_n(s)z$$

$$= \sum_{n=1}^{\infty} CU_n(t,s)P_n(s)z$$

$$= CU(t, s)\, z.$$

Next, we show that U(t, s) is strongly continuous in $[0, \infty)$. In fact:

$$\|U(t, s)z - Cz\|^2 = \left\| \sum_{n=1}^{\infty} U_n(t,s)P_n(s)z - \sum_{n=1}^{\infty} P_n(s)Cz \right\|^2$$

$$= \sum_{n=1}^{\infty} \|(U_n(t,s) - C)P_n(s)z\|^2$$

$$= \sum_{n=1}^{\infty} \|(U_n(t,s) - C)P_n(s)z\|^2 + \sum_{n=N+1}^{\infty} \|(U_n(t,s) - C)P_n(s)z\|^2$$

$$\leq \sup_{1 \leq n \leq N} \|(U_n(t,s) - C)P_n(s)z\|^2\, N + K \sum_{n=N+1}^{\infty} \|P_n(s)z\|^2$$

where $$K = \sup_{0 \leq s \leq t\, 1;\, n \geq 1} \|U_n(t,s) - C\|^2 \leq (g(t,s)+1)^2$$

Since $\{U_n(t, s)\}_{0 \leq s \leq t \leq T}$ (n = 1, 2, . . .) is a strongly continuous regularized evolution operator and $\{P_n(s)\}_{n \geq 1}$ is a complete orthogonal projections, given an arbitrary $\epsilon > 0$ we have, for some natural number N and $0 < s < t < 1$, the following estimates:

$$\sum_{n=N+1}^{\infty} \|P_n(s)z\|^2 < \frac{\epsilon}{2K}, \quad \sup_{1 \leq n \leq N} \|U_n(t,s) - C)P_n(s)z\|^2 < \frac{\epsilon}{2N}$$

and hence

$$\|U(t, s)\, z - Cz\|^2 < \frac{\epsilon}{2N} N + K \frac{\epsilon}{2K}, = \epsilon\, .$$

Therefore, U(t, s) is a strongly continuous regularized evolution operator in $[0, \infty)$. Let A(t) be the generator of this evolution operator. Then, from definition 3.4, we have for all $z \in D$,

$$A(t)\, z = C^{-1} \lim_{h \to 0^+} \frac{U(t+h,t)z - Cz}{h} = C^{-1} \lim_{h \to 0^+} \sum_{n=1}^{\infty} \frac{(U_n(t+h,t) - C)}{h} P_n(t)z.$$

Therefore,

$$P_m(t)A(t)z = P_m(t)\left(C^{-1} \lim_{h \to 0^+} \sum_{n=1}^{\infty} \frac{(U_n(t+h,t) - C}{h} P_n(t)z \right)$$

$$= C^{-1} \lim_{h \to 0^+} \frac{(U_m(t+h,t) - C)}{h} P_m(t)z$$

$$= A_m(t)\, P_m(t)z.$$

Hence,

$$A(t)z = \sum_{n=1}^{\infty} P_n(t)A(t)z = \sum_{n=1}^{\infty} A_n(t)P_n(t)z,$$

and

$$D \subset W = \left\{ z \in Z : \sum_{n=1}^{\infty} \| A_n(t)P_n(t)z \|^2 \; \infty, \forall t \in [0, \infty) \right\}$$

Now, suppose A(t) is closed and $z \in \left\{ z \in Z : \sum_{k=1}^{\infty} \| A_k(t)P_k(t)z \|^2 < \infty, \forall t \in [0, \infty) \right\}$.

Then

$$\sum_{k=1}^{\infty} \| A_k(t)P_k(t)z \|^2 < \infty, \; t \in [0, \infty)$$

and

$$y = \sum_{k=1}^{\infty} A_k(t)P_k(t)z.$$

Therefore, if we consider $z_n = \sum_{k=1}^{n} P_k(t)z$, then $z_n \in D$ and $A(t)z_n = \sum_{k=1}^{n} A_k(t)P_k(t)z$.

Hence, $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty} A(t)z_n = y$ and since A(t) is a closed linear operator we get that $z \in D$ and $A(t)\, z = y$.

(iii) If A(t) is a closed operator, then D = W. Then for all $z \in D$ we consider the following estimate

$$\sum_{n=1}^{\infty} \| A_n(t)P_n(t)z \|^2 < \infty, \text{ for all } 0 \le t < \infty. \text{ Then}$$

$$\begin{aligned}
\left\| \sum_{n=1}^{\infty} A_n(t)P_nU(t,s)z \right\|^2 &= \left\| \sum_{n=1}^{\infty} A_n(t)P_n(t) \left(\sum_{k=1}^{\infty} U_k(t,s)P_k(s)z \right) \right\|^2 \\
&= \left\| \sum_{n=1}^{\infty} A_n(t)U_n(t,s)P_n(s)z \right\|^2 \\
&\le \sum_{n=1}^{\infty} \| A_n(t)U_n(t,s)P_n(s)z \|^2 \\
&\le (g(t,s))^2 \sum_{n=1}^{\infty} \| A_n(t)P_n(s)z \|^2 \\
&< \infty.
\end{aligned}$$

for all $0 \le s \le t < \infty$. Hence, $U(t, s)\, z \in D$ for all $z \in D$ and $0 \le s \le t < \infty$.

5. APPLICATIONS

Here we shall use the foregoing results to find a formula for the following system of non autonomous partial parabolic equations:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} & = & D\Delta u(t,s) + B(t)u(t,s) + \alpha^2(x)\, f(t,x), t > 0, u \in \mathbb{R}^n, \\ \dfrac{\partial u(t,x)}{\partial \eta} & = & 0, t > x \in \ \partial\Omega, \\ u(0,x) & = & \alpha^2(x)\, \psi(x), x \in \Omega, \end{cases} \tag{5.1}$$

. . . . . Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$, α, $\psi\, L^2(\Omega)$, with $\alpha(x) > 0, \forall x \in \Omega$ and D is an n × n matrix whose eigenvalues are semisimple with non negative real part or strictly positive and $f : \mathbb{R} \times \Omega \to \mathbb{R}^n$ is a smooth function. We assume that the operator $B \in P_\infty([0, \infty); \Gamma(Z))$:

$$\left\{ B / \langle z_1, B(\cdot) z_2 \rangle \text{es medible } \forall z_1 z_2 \in Z \text{ y } (\text{ess} \sup_{0 \leq t < \infty} \| B(t) \|_{\Gamma(z)} < \infty \right\}$$

with $Z = L^2(\Omega)$.

5.1. Abstract Formulation of the Problem

We choose a Hilbert space where system (5.1) can be written as an abstract differential equation; to this end, we consider the following notations:

Let us consider the Hilbert space $Z = L^2(\Omega, \mathbb{R})$ y $0 = \lambda_1 < \lambda_2 < \ldots < \lambda_j \to \infty$ the eigenvalues of each one with .finite multiplicity $-\Delta$, equal to the dimension of the corresponding eigenspace. Then, we have the following well known properties (see [9]):

(i) There exists a complete orthonormal set $\{\phi_{j,k}\}$ of eigenvectors of $-\Delta$.

(ii) For all $\xi \in D(-\Delta)$ we have

$$-D\xi = \sum_{j=1}^{\infty} \lambda_j \frac{\gamma_j}{j=1} < \xi, \phi_{j,k} = \sum_{j=1}^{\infty} \lambda_j E_j \xi, \tag{5.2}$$

where

$$E_j x = \sum_{k=1}^{\gamma_j} < \xi, \phi_{j,k} > \phi_{j,k}. \tag{5.3}$$

So, $\{E_j\}$ is a family of complete orthogonal projections in H and

(iii) Δ generates an analytic semigroup $\{T_\Delta(t)\}$ given by

$$T_\Delta(t)x = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j \xi \tag{5.4}$$

(iv) There exists a constant $M \geq 1$ such that:

$$\| e^{-\lambda_j Dt} \| \leq M, \quad t \leq 0, \quad n = 1, 2, 3, \ldots$$

Now, we denote by Z the Hilbert space $(L^2(\Omega, \mathbb{R}^n))$ and define the following operator

$$A : D(A) \subset Z \to Z,\ A\psi = -D\Delta\psi$$

with $D(A) = H^2(\Omega, \mathbb{R}^n) \cap H_0^1(\Omega, \mathbb{R}^n)$. Therefore, for all $z \in D(A)$ we obtain

$$Az = \sum_{j=1}^{\infty} \lambda_j D P_j z \quad \text{and} \quad \sum_{j=1}^{\infty} P_j z, \| z \|^2 = \sum_{j=1}^{\infty} \| P_j z \|^2,\ z \in Z,$$

where

$$P_j = \operatorname{diag}(E_j, E_j, \ldots, E_j)$$

is a family of complete orthogonal projections in Z. Using Lemma 3.1 from [4] one can prove the following theorem.

Theorem 5.1: The operator $-A$ is the infinitesimal generator of a strongly continuous semigroup $\{T_A(t)\}_{t \geq 0}$ in the space Z, given by

$$T_A(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j D t} P_j z,\ z \in Z,\ t \geq 0. \tag{5.5}$$

Consequently, the system (5.1) can be written as an abstract ordinary differential equation in Z:

$$\begin{cases} \dfrac{dz(t)}{dt} = -Az(t) + B(t)z(t) + C^2 f^e(t), & t > 0 \\ z(0) = C^2 z_0, \end{cases} \tag{5.6}$$

where $z_0 \in Z$, $C : Z \to Z$ is an injective bounded linear operator defined by :

$$(Cz)(x) = \alpha(x)\, z(x),\ z \in Z\ x \in \Omega,$$

which is trivially injective, and $f^e : [0, \infty) \to Z$ is a function defined as follows:

$$f^e(t)(x) = f(t, x),\ t \geq 0,\ x \in \Omega.$$

In case that $f^e \equiv 0$ the system (5.6) is given by:

$$\begin{cases} \dfrac{dz(t)}{dt} = -Az(t) + B(t)z(t), & t > 0, \\ z(s) = C^2 z_0 \end{cases} \tag{5.7}$$

Definition 5.2: (Mild Solution) Note any solution z of the initial value problem (5.7) satisfies the integral equation.

$$z(t) = T_A(t-s)\, Cz_0 + \int_s^t T_A(t-\gamma)\, B(\gamma) z(\gamma) d\gamma,\ t \in [s, \infty), \tag{5.8}$$

but the converse is not true since a solution of (5.8) is not nessessarily differential. We shall refer to a continuous solution of (5.8) as a mild solution of (5.7); a mild solution is thus a kind of generalized solution.

Define the following operator in the space Z for $t \geq s \geq 0$ by

$$U(t, s)\, z_0 = T_A(t-s)\, Cz_0 + \int_s^t T_A(t-\gamma)\, B(\gamma)\, U(t, \gamma) z_0 d\gamma. \tag{5.9}$$

Then, Lemma 4.1 we get the following results, Theorem 5.3: The family of operators $\{U(t, s)\}_{t \geq s \geq 0}$ defined by (5.9) is a strongly continous evolution opeator on Z such that

$$U(t, s)\, z = \sum_{n=1}^{\infty} U_n(t,s) P_n z\ ,\ z \in Z,\ t \geq s \geq 0, \tag{5.10}$$

where $\{\{Uj(t, s)\}_{t \geq s \geq 0,}\ j = 1, 2, 3, ...\}$ is a family of strongly continuous evolution operators on $Z^j = P_j Z$ defined as follows

$$U_j(t, s)\, z_0^j = z^j\,(t, s, z_0^j)$$

where $z^j(\cdot)$ is the unique solution of the initial value problem

$$\begin{cases} \dfrac{dz(t)}{dt} = -\lambda_j D z(t) + B_j(t) z(t),\ \ t > s, \\ z(s) = C z_0^j \end{cases} \tag{5.11}$$

Therefore, the system (5.7) y (5.6) are equivalent to the following two system of ordinary differential equations in Z respectively.

$$\begin{cases} \dfrac{dz(t)}{dt} = -\Lambda(t) + z(t), \quad\ t > s, \\ z(s) = C z_0, \end{cases} \tag{5.12}$$

$$\begin{cases} \dfrac{dz(t)}{dt} = \Lambda(t) + z(t) + f^{e}(t), \quad t > s, \\ z(s) = C z_0, \end{cases} \tag{5.13}$$

where $\Lambda(t) = -A + B(t)$ is the infinitesimal generator of the evolution operator $\{U(t, s)\}_{t \geq s}$.

Finally, applying Theorem 5.3 we obtain the following result:

Theorem 5.4: The abstract Cauchy problem in the Hilbert space Z

$$\begin{cases} \dfrac{dz(t)}{dt} = \Lambda(t) + z(t) + C^2 f^{e}(t), \quad t > s, \\ z(s) = C z_0, \end{cases}$$

where $\Lambda(t)$ is the generator of the evolution operator $\{U(t, s)\}_{t \leq s}$ admits one and only one solution given by:

$$z(t) = U(t,s) z_0 + \int_s^t U(t,\gamma)\, C f^{e}(\gamma) d\gamma,\ t \geq s. \tag{5.14}$$

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