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Approximate Controllability of Linear Cascade Systems in $H = L^2(\Omega)$

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Abstract: In this paper we study the interior approximate controllability of the following cascade systems of m coupled evolution equations in the Hilbert space $H = L^2(\Omega)$

$$\begin{cases} \dot{z}_1 = -Az_1 + 1_\omega u(t), t \in (0, \tau], \quad \tau > 0 \\ \dot{z}_i = -Az_i + z_{i-1}, \quad i = 2,3, ..., m, \end{cases}$$

where Ω is a bounded in $\mathbb{R}^N (N \geq 1)$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω, control u belongs to $L^2(0, \tau; L^2(\Omega))$ and the operator $A : D(A) \subset H \to H$ is an unbounded linear operator with the spectral decomposition $A\xi = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} < \xi, \phi_{j,k} > \phi_{j,k} \ > \ \phi_{j,k}$ with $0 < \lambda_1 < \lambda_2 < .... < \lambda_j < ... \lambda_n \to \infty$ and $\{\phi_{j,k}\}$ is a complete orthonormal set of analytic functions in H. Under these conditions we prove the following statement: The system is approximately controllable on $[0, \tau]$. Moreover, we exhibit a sequence of controls steering the system from an initial state to a $\in$-neighborhood of final state in a prefixed time τ. Our result can be apply to the semilinear *nD* heat equation, the Ornstein-Uhlenbeck equation, the Laguerre equation, the Jacobi equation, amount others.

Key words and phrases. strongly continuous semigroups, controllability, cascade systems .

1. INTRODUCTION

This work has been motivated by the work done in [6], [10], [16] and [1], where the authors study, respectively, the controllability of one dimensional coupled of degenerated parabolic equations, the null controllability of a cascaded system of parabolic-hyperbolic equations and the exact controllability of a cascade system of conservative equations in Hilbert spaces with applications to the Schrödinger equations, the wave equation and the heat equation. Models of cascade systems can be found in mathematical biology, chemistry, engineering and in a broad variety of physical situations(see [9],[12],[15]). A simple model of this kind of systems appears in mixing problems: Suppose that three tanks connected contains each one 100gls of solution of a certain chemical. Starting at a certain instant a solution of the same chemical, with concentration $u(t)$ *lb*/gal, is allowed to flows in to the first tank at the rate of *R gl*/*mi*. The mixture is drained off at the same rate into the second tank; from the second tank in to the third tank chemical flows at the same rate and the solution flows aut of this tank a the same rate. This problem can be formulated as a cascade control system for the amount of chemical in these three tanks at the time t. In fact, the three tanks can be denoted respectively by T_1, T_2 and T_3 and $u(t)$ the concentration of chemical flowing in to T_1 acts as a control, $z_i(t)$, $i = 1, 2, 3$ the amount of chemical in T_i at time t and $\frac{z_i}{100}$ the concentration of the chemical in T_i at time t. Therefore, the variation of the chemical in each tanks is given by the following cascade system

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$$\begin{cases} \dfrac{dz_1(t)}{dt} = Ru(t) - R\dfrac{q_1(t)}{100}, \\ \dfrac{dz_2(t)}{dt} = R\dfrac{z_1(t)}{100} - R\dfrac{z_2(t)}{100}, \\ \dfrac{dz_3(t)}{dt} = R\dfrac{z_2(t)}{100} - R\dfrac{z_3(t)}{100}. \end{cases} \tag{1.1}$$

Now, if we put $a = R$ and $b = R/100$, this system can be written as follows

$$\begin{cases} z_1'(t) = au(t) - bz_1(t) \\ z_2'(t) = bz_1(t) - bz_2(t) \\ z_3'(t) = bz_2(t) - bz_3(t) \end{cases} \tag{1.2}$$

and

$$z' = Az + Bu(t), \tag{1.3}$$

where

$$A = \begin{pmatrix} -b & 0 & 0 \\ b & -b & 0 \\ 0 & b & -b \end{pmatrix}, \quad B = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \text{ and } z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}.$$

It is well known (see [13]) that the system (1.3) is controllable if and only if

$$Rank[B\,|AB\,|A^2B] = 3$$

which, doing the respective calculation, is trivially true. In consequence, we have proved that the system (1.3) is controllable.

In this paper we shall study the interior approximate controllability of the following cascade system of m coupled evolution equations in the Hilbert space $H = L^2(\Omega)$

$$\begin{cases} \dot{z}_1 = -Az_1 + 1_w u(t), \ t \in (0, \tau)], \ \tau > 0 \\ \dot{z}_2 = -Az_2 + z_1, \\ \dot{z}_3 = -Az_3 + z_2, \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ \dot{z}_m = -Az_m + z_{m-1}, \end{cases} \tag{1.4}$$

where Ω is a bounded in $\mathbb{R}^N (N \geq 1), \omega$ is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the ω, control u belongs to $L^2(0, \tau; U)$ with $U = H = L^2(\Omega)$ and the operator $A : D(A) \subset H \to H$ is an unbounded linear operator with the spectral decomposition

$$A\xi = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} < \xi, \phi_{j,k} > \phi_{j,k} = \sum_{j=1}^{\infty} \lambda_j E_j \xi, \tag{1.5}$$

with $<\cdot, \cdot>$ denoting an inner product in H, and

The eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots \lambda_n \to \infty$ of A have finite multiplicity γ_j equal to the dimension of the corresponding eigenspace, and $\{\phi_{j,k}\}$ is a complete orthonormal set of analytic eigenfunctions (eigenvectors) of A. So, $\{E_j\}$ is a complete family of orthogonal projections in H and $\xi = \sum_{j=1}^{\infty} E_j \xi, \xi \in H.$ The operator $-A$ generates a strongly continuous compact semigroup $\{T(t)\}$ given by

$$T(t)\xi = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j \xi. \tag{1.6}$$

Without lose of generality, we shall assume that $m = 3$; that is to say, it is enough to study the particular system of three coupled evolution equations in H

$$\begin{cases} \dot{z}_1 = -Az_1 + 1_\omega u(t), & t \in (0, \tau], \quad \tau > 0 \\ \dot{z}_2 = -Az_2 + z_1, \\ \dot{z}_3 = -Az_3 + z_2. \end{cases} \tag{1.7}$$

Our technique is different than the one used in the above references, it is base on the following basis results from Analytic Functions Theory

Theorem 1.1: (see Theorem 1.23 from [3], pg. 20) Suppose $\Omega \subset \mathbb{R}^n$ is open, non-empty and connected set, and f is real analytic function in Ω with $f = 0$ on a non-empty open subset ω of Ω. Then, $f = 0$ in Ω.

As an application we can prove the interior approximate controllability of the following cascade system of three coupled parabolic equations

$$\begin{cases} v_t(t,x) = \Delta v(t,x) + 1_\omega u(t,x), & \text{in } (0,\tau]\times\Omega, \\ w_t(t,x) = \Delta w(t,x) + v(t,x), & \text{in } (0,\tau]\times\Omega, \\ y_t(t,x) = \Delta y(t,x) + w(t,x), & \text{in } (0,\tau]\times\Omega, \\ v = w = y = 0, & \text{on } (0,\tau)\times\partial\Omega, \\ v(0,x) = v_0(x),\ w(0,x) = w_0(x),\ y(0,x) = y_0(x), & x \in \Omega, \end{cases} \tag{1.8}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$, $z_0 \in L^2(\Omega)$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω, the distributed control u belong to $\in L^2([0, \tau], L^2(\Omega))$.

2. MAIN RESULT

In this section we shall prove the main result of this work; to this end, we notice that for arbitrary $z_0 = (z_{01}, z_{02}, z_{03})^T \in Z = H = H \times H$ and $u \in L^2([0, \tau]; U)$ the system admits only one mild solution $z(t) = (z_1(t), z_2(t), z_3(t))^T \in Z$ given by

$$z_1(t) = T(t)z_{01} + \int_0^t T(t-s)B_\omega u(s)ds, \ t \in [0, \tau], \tag{2.1}$$

$$z_2(t) = T(t)z_{02} + \int_0^t T(t-s)z_1(s)ds, \quad t \in [0, \tau], \tag{2.2}$$

$$z_3(t) = T(t)z_{03} + \int_0^t T(t-s)z_2(s)ds, \quad t \in [0, \tau], \tag{2.3}$$

where $B_\omega : H \to H$ is the bounded linear operator given by $B_\omega f = 1_\omega f$.

Definition 2.1: (Approximate Controllability) The system (1.7) is said to be approximately controllable on $[0, \tau]$ if for every $z_0, z_1 \in Z, \in > 0$ there exists $u \in L^2(0, \tau; U)$ such that the mild solution $z(t)$ of (1.7) corresponding to u verifies:

$z(0) = z_0$ and $\|z(\tau) - z_1\| < \varepsilon$.

Definition 2.2: For the system (1.7) we define the following concept: The controllability map (for $\tau > 0$) $G : L^2(0, \tau; U) \to Z$ is given by $Gu = (G_1u, G_2u, G_3u)^T$ where G_1, G_2,G3 : L2(0, ƒÑ ;U) .¨ H are operators define as follows

$$G_1u = \int_0^\tau T(\tau - s) B_\omega u(s) ds, \tag{2.4}$$

$$G_2u = \int_0^\tau T(\tau - s)\, z_1(s) ds = \int_0^\tau \int_0^8 T(\tau - \alpha) B_\omega u(\alpha)\, d\alpha, ds, \tag{2.5}$$

$$G_3u = \int_0^\tau T(\tau - 8) z_2(s) ds = \int_0^\tau \int_0^8 \int_0^\alpha T(\tau - \beta) B_\omega u(\beta)\, d\beta d\alpha ds. \tag{2.6}$$

Now, using Fubini'fs Theorem, these operators can be written as follows:

$$G_1u = \int_0^\tau T(\tau - s) B_\omega u(s) ds, \tag{2.7}$$

$$G_2u = \int_0^\tau (\tau - \alpha) T(\tau - \alpha) B_\omega u(\alpha) d\alpha, \tag{2.8}$$

$$G_3u = \int_0^\tau \frac{(\tau - \beta)^2}{2} T(\tau - \beta) B_\omega u(\beta) d\beta \tag{2.9}$$

whose adjoint operators G_1^*, G_2^*, $G_3^* : H \to L^2(0, \tau; U)$ are

$$(G_1^* z_1)(s) = B_\omega^* T^*(\tau - s) z_1, \ \forall s \in [0, \tau], \forall z_1 \in H, \tag{2.10}$$

$$(G_2^* z_2)(s) = (\tau - s) B_\omega^* T^*(\tau - s) z_2, \quad \forall s \in [0, \tau], \quad \forall z_2 \in H,$$

$$(G_3^* z_3)(s) = \frac{(\tau - s)^2}{2} B_\omega^* T^*(\tau - s) z_3, \forall s \in [0, \tau], \forall z_3 \in H. \tag{2.12}$$

Therefore, the adjoint operator $G^* : Z \to L^2(0, \tau; U)$ of G is given by the formula

$$G^*Z = G^* \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = G_1^* z_1 + G_2^* z_2 + G_3^* z_3 \tag{2.13}$$

$$= B_\omega^* T^*(\tau - \cdot) \left\{ z_1 + (\tau - \cdot) z_2 + \frac{(\tau - \cdot)^2}{2} z_3 \right\}.$$

The following lemma holds in general for a linear bounded operator $G : W \to Z$ between Hilbert spaces W and Z.

Lemma 2.3: (see [7], [8], [2] and [14]) The equation (1.7) is approximately controllable on $[0, \tau]$ if and only if one of the following statements holds:

(a) $\overline{\text{Rang}(G)} = Z$.

(b) $Ker(G^*) = \{0\}$.

(*c*) $\langle GG^* z, z\rangle > 0$, $z \neq 0$ in Z.

(*d*) $\lim_{\alpha\to 0^+} \alpha(\alpha I + GG^*)^{-1} z = 0$.

(*e*) $\sup_{\alpha>0} \| \alpha(\alpha I + GG^*)^{-1} \| \leq 1$.

(*f*) $B_\omega^* T^*(\tau - s)\left\{z_1 + (\tau - s)z_2 + \dfrac{(\tau - s)^2}{2} z_3\right\} = 0, \ \forall s \in [0, \tau], \ \Rightarrow z_1 = z_2 = z_3 = 0.$

(*g*) For all $z \in Z$ we have $Gu_\alpha = z - \alpha(\alpha I + GG^*)^{-1}z$, where

$$u_\alpha = G^*(\alpha I + GG^*)^{-1} z, \quad \alpha \in (0,1].$$

So, $\lim_{\alpha\to 0} Gu = z$ and the error $E_\alpha z$ of this approximation is given by the formula

$$E_\alpha z = \alpha(\alpha I + GG^*)^{-1} z, \quad \alpha \in (0,1].$$

Remark 2.4: The Lemma 2.3 implies that the family of linear operators $\Gamma_\alpha : Z \to L^2(0,\tau;U)$, defined for $0 < \alpha \leq 1$ by

$$\Gamma_\alpha z = G^*(\alpha I + GG^*)^{-1} z, \tag{2.14}$$

is an approximate inverse for the right of the operator G, in the sense that

$$\lim_{\alpha\to 0} G\Gamma_\alpha = I. \tag{2.15}$$

Theorem 2.5: The system (1.7) is approximately controllable on [0, τ]. Moreover, a sequence of controls steering the system (1.7) from initial state z_0 to an $\in$ neighborhood of the final state z_1 at time $\tau > 0$ is given by

$$u_\alpha(t) = G^*(\alpha I + GG^*)^{-1}(z_1 - \mathcal{T}(\tau)z_0), \ \alpha \in (0,1],$$

and the error of this approximation E_α is given by

$$E_\alpha = \alpha(\alpha I + GG^*)^{-1}(z_1 - \mathcal{T}(\tau)z_0)$$

where

$$\mathcal{T}(t) = \begin{pmatrix} T(t) \\ T(t) \\ T(t) \end{pmatrix}.$$

Proof: We shall apply Lemma 2.3.*f* to prove the controllability of system (1.7). To this end, we observe that $T^*(t)\xi = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j \xi$ and $B_\omega^* = B_\omega$. Then,

$$(G^* z)(t) = B_\omega^* T^*(\tau - t)\,(z_1 + (\tau - t)z_2 + \frac{(\tau - t)^2}{2} z_3) = 0, \quad \forall t \in [0, \tau].$$

i.e.,

$$\sum_{j=1}^{\infty} e^{-\lambda_j t}(B_\omega^* E_j z_1 + tB_\omega^* E_j z_2 + \frac{t^2}{2} B_\omega^* E_j z_3) = 0, \quad \forall t \in [0, \tau].$$

Now, using the fact that $\lim_{t\to\infty} t^m e^{-\lambda_j t} = 0$ and some ideas from the proof of Lemma 3.14 in [7], pg. 62, we obtain that

$$B_\omega^* E_j z_1 + t B_\omega^* E_j z_2 + \frac{t^2}{2} B_\omega^* E_j z_3 = 0, \quad j = 1, 2, 3, \ldots.$$

Hence,

$$B_\omega^* E_j z_1 = B_\omega^* E_j z_2 = B_\omega^* E_j z_3 = 0, \quad j = 1, 2, 3, \ldots.$$

In other words,

$$\sum_{k=1}^{\gamma_j} < z_i, \phi_{j,k} > 1_\omega \phi_{j,k} = 0, \quad j = 1, 2, 3, \ldots; i = 1,2,3.$$

i.e.,

$$\sum_{k=1}^{\gamma_j} < z_i, \phi_{j,k} > \phi_{j,k} = 0, \quad \forall x \in \omega, j = 1, 2, 3, \ldots; i = 1,2,3.$$

Since, by hypothesis $\{\phi_{j,k}\}$ is a complete orthonormal set of analytic functions in H, then applying Theorem 1.1 we obtain that

$$\sum_{k=1}^{\gamma_j} < z_i, \phi_{j,k} > \phi_{j,k} = 0, \quad \forall x \in \Omega, j = 1, 2, 3, \ldots; i = 1,2,3,$$

and consequently

$$< z, \phi_{j,k} > = 0 \;\; j = 1, 2, \ldots \text{ and } \; k = 1, 2, \ldots \gamma_j.$$

Which implies that $z = 0$, and the system (1.7) is approximately controllable on $[0, \tau]$.

Now, given the initial and the final states z_0 and z_1, we consider the sequence of controls

$$\begin{aligned} u_\alpha(\cdot) &= G^*(\alpha I + GG^*)^{-1}(z_1 - \mathcal{T}(\tau)z_0) \\ &= G^*(\alpha I + GG^*)^{-1}(z_1 - \mathcal{T}(\tau)z_0), \quad \alpha > 0. \end{aligned}$$

Then,

$$\begin{aligned} Gu_\alpha &= GG^*(\alpha I + GG^*)^{-1}(z_1 - \mathcal{T}(\tau)z_0) \\ &= (\alpha I + GG^* - \alpha I)(\alpha I + GG^*)^{-1}(z_1 - \mathcal{T}(\tau)z_0) \\ &= z_1 - e^{\alpha\tau} T(\tau) z_{20} - \alpha(\alpha I + GG^*)^{-1}(z_1 - \mathcal{T}(\tau)z_{20}). \end{aligned}$$

From Lemma 2.3.c we know that

$$\lim_{\alpha \to 0+} \alpha(\alpha I + GG^*)^{-1}(z_1 - \mathcal{T}(\tau)z_0) = 0.$$

Therefore,

$$\lim_{\alpha \to 0+} Gu_\alpha = z_1 - \mathcal{T}(\tau)z_0.$$

i.e.,

$$z_{11} = \lim_{\alpha \to 0+} \left\{ T(\tau)z_{01} + \int_0^\tau T(\tau - s) B_\omega u_\alpha(s) ds \right\}, \tag{2.16}$$

$$z_{12} = \lim_{\alpha \to 0+} \left\{ T(\tau)z_{02} + \int_0^\tau (\tau - s) T(\tau - s) B_\omega u_\alpha(s) ds \right\}, \tag{2.17}$$

$$z_{13} = \lim_{\alpha\to 0+}\{T(\tau)z_{03} + \int_0^\tau \frac{(\tau-s)^2}{2} T(\tau-s)B_\omega u_\alpha(s)ds\}, \tag{2.18}$$

where

$$z_1 = \begin{pmatrix} z_{11} \\ z_{12} \\ z_{13} \end{pmatrix} \text{ and } z_0 = \begin{pmatrix} z_{01} \\ z_{02} \\ z_{03} \end{pmatrix}.$$

This completes the proof of the Theorem.

3. APPLICATIONS

As an application we can prove the interior approximate controllability of the linear cascade system of three coupled parabolic equations (1.8). To this end, we shall need the following Theorem

Theorem 3.1: (See [14]) The eigen functions of the operator $-\Delta$ with Dirichlet boundary conditions on Ω are real analytic functions in Ω.

Now, we describe the space in which this problem will be situated as an abstract semilinear cascade system of ordinary differential equations. Let us consider $H = L^2(\Omega)$ and the linear unbounded operator

$A : D(A) \subset H \to H$ defined by $A\phi = -\Delta\phi$, where

$$D(A) = H_0^1(\Omega) \cap H^2(\Omega). \tag{3.1}$$

The operator A has the following very well known properties: The spectrum of A consists of only eigen values

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j \to \infty,$$

each one with multiplicity γ_j equal to the dimension of the corresponding eigenspace.

(*a*) There exists a complete orthonormal set $\{\phi_{j,k}\}$ of analytic eigenfunctions of A.

(*b*) For all $\xi \in D(A)$ we have

$$A\xi = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} < \xi, \phi_{j,k} > \phi_{j,k} = \sum_{j=1}^{\infty} \lambda_j E_j \xi, \tag{3.2}$$

where $< \cdot, \cdot >$ is the inner product in H. The system (1.8) can be written as an abstract system of ordinary differential equations in the space $H = L^2(\Omega)$

$$\begin{cases} \dot{v} = -Av + 1\omega u(t), \quad t \in (0, \tau], \\ \dot{w} = -Aw + v, \\ \dot{y} = -Ay + \omega, \\ v(0) = v_0, \; w(0) = w_0, \; y(0) = y_0, \end{cases} \tag{3.3}$$

where the control function u belongs to $L^2(0, \tau; H)$.

Theorem 3.2: For all open non-empty set $\omega \subset \Omega$ and $\tau > 0$ the system (3.3) is approximately controllable on $[0, \tau]$.

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