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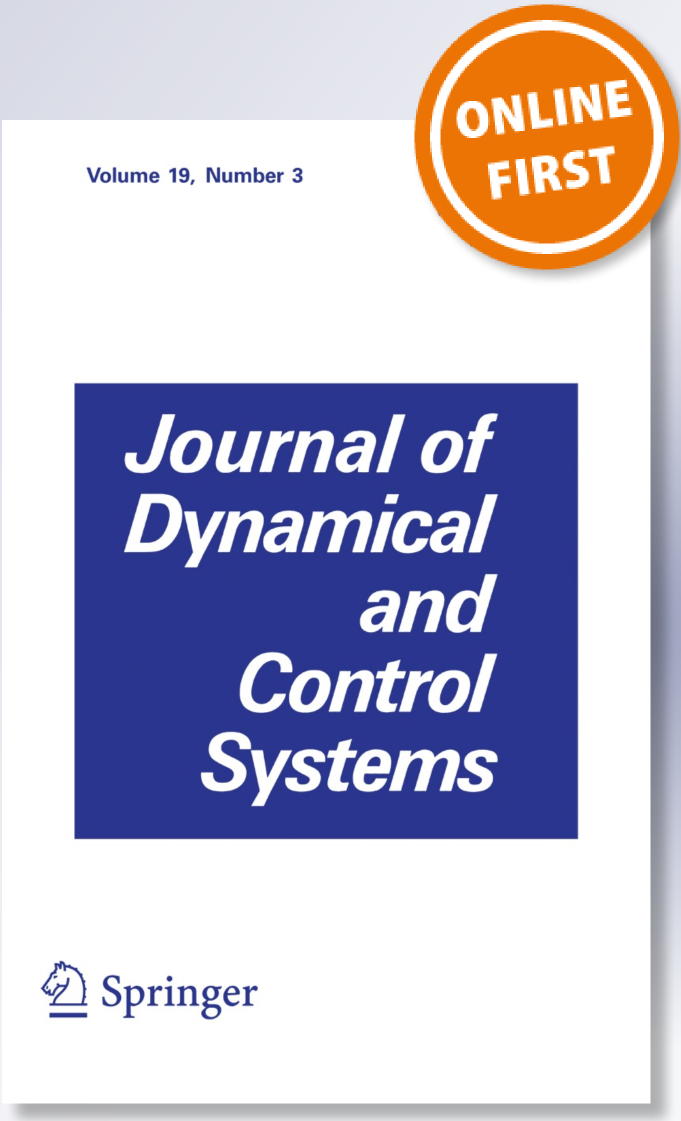

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Controllability of the Semilinear Beam Equation

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Abstract In this paper, we prove the approximate controllability of the following semilinear beam equation:

$$\begin{cases} \dfrac{\partial^2 y(t,x)}{\partial t^2} = 2\beta\Delta\dfrac{\partial y(t,x)}{\partial t} - \Delta^2 y(t,x) + u(t,x) + f(t,y,y_t,u), \text{ in } (0,\tau)\times\Omega, \\ y(t,x) = \Delta y(t,x) = 0, \quad \text{on } (0,\tau)\times\partial\Omega, \\ y(0,x) = y_0(x), \quad y_t(x) = v_0(x), x\in\Omega, \end{cases}$$

in the states space $Z_1 = D(\Delta)\times L^2(\Omega)$ with the graph norm, where $\beta > 1$, Ω is a sufficiently regular bounded domain in $I\!R^N$, the distributed control u belongs to $L^2([0,\tau];U)$ $(U = L^2(\Omega))$, and the nonlinear function $f:[0,\tau]\times I\!R\times I\!R\times I\!R \longrightarrow I\!R$ is smooth enough and there are $a, c \in I\!R$ such that $a < \lambda_1^2$ and

$$\sup_{(t,y,v,u)\in Q_\tau} | f(t,y,v,u) - ay - cu | < \infty,$$

where $Q_\tau = [0,\tau]\times I\!R\times I\!R\times I\!R$. We prove that for all $\tau > 0$, this system is approximately controllable on $[0,\tau]$.

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1 Introduction

In this paper, we prove the approximate controllability of the following semilinear beam equation:

$$\begin{cases} \dfrac{\partial^2 y(t,x)}{\partial t^2} = 2\beta\Delta\dfrac{\partial y(t,x)}{\partial t} - \Delta^2 y(t,x) + u(t,x) \\ \qquad\qquad + f(t, y(t,x), y_t(t,x), u(t,x)), \quad \text{in } (0,\tau)\times\Omega \\ y(t,x) = \Delta y(t,x) = 0, \quad \text{on } (0,\tau)\times\partial\Omega, \\ y(0,x) = y_0(x), \quad y_t(x) = v_0(x), \quad x \in \Omega, \end{cases} \tag{1.1}$$

where $\beta > 1$, Ω is a sufficiently regular bounded domain in $I\!R^N$, the distributed control u belongs to $L^2([0,\tau];U)$ $(U = L^2(\Omega))$, and the nonlinear function $f : [0,\tau]\times I\!R\times I\!R\times I\!R \longrightarrow I\!R$ is smooth enough and there are $a, c \in I\!R$ such that $a < \lambda_1^2$, with λ_1 as the first eigenvalue of $-\Delta$ with Dirichlet boundary condition, and

$$\sup_{(t,y,v,u)\in Q_\tau} | f(t,y,v,u) - ay - cu | < \infty, \tag{1.2}$$

where $Q_\tau = [0,\tau]\times I\!R\times I\!R\times I\!R$. We prove that for all $\tau > 0$, system (1.1) is approximately controllable on $[0,\tau]$.

Here, the state space is $Z_1 = \left[H^2(\Omega)\bigcap H_0^1(\Omega)\right]\times L^2(\Omega) = D(-\Delta)\times L^2(\Omega)$ endowed with the graph norm; that is to say

$$\left\| \begin{pmatrix} y \\ v \end{pmatrix} \right\|_{Z_1} = \sqrt{\|(-\Delta)y\|_{L^2}^2 + \|v\|_{L^2}^2},$$

where

$$\|v\| = \|v\|_{L^2} = \sqrt{\int_\Omega |v(x)|^2 dx}, \quad \forall v \in L^2(\Omega).$$

Remark 1.1 The term $-2\beta\Delta\dfrac{\partial y(t,x)}{\partial t}$ in Eq. (1.1) acts as a damping force; hence, the energy space used to set up the wave equation is not adequate here. In fact, the uncontrolled linear equation can be transformed into a system of parabolic equations of the form $z_t = D\Delta z$ (see [4]), which shows that $Z_1 = \left[H^2(\Omega)\bigcap H_0^1(\Omega)\right]\times L^2(\Omega)$ is the right space for an abstract formulation of the problem. The controllability of similar equations was proposed in [8] as a future work.

Definition 1.1 (Approximate Controllability) System (1.1) is said to be approximate controllable on $[0,\tau]$ if for every $z_0 = (y_0, v_0)^T$, $z_1 = (y_1, v_1)^T \in Z_1$, $\epsilon > 0$, there

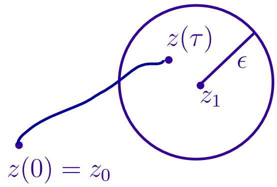

Fig. 1 Verification of $\|z(\tau) - z_1\|$

exists $u \in L^2(0, \tau; U)$ such that the mild solution $z(t) = (y(t), y_t(t))^T$ of Eq. (1.1), corresponding to u, verifies (Fig. 1) the following:

$$z(0) = z_0 \quad \text{and} \quad \| z(\tau) - z_1 \|_{Z_1} < \epsilon.$$

Under certain conditions, we prove the following statement: The nonlinear system (1.1) is approximately controllable on $[0, \tau]$. Moreover, we can exhibit a sequence of controls steering the system from an initial state to an ϵ-neighborhood of a final state in a prefixed time τ.

Now, we shall describe the strategy of this work:

First, we prove that the auxiliary linear system in $(0, \tau) \times \Omega$

$$\begin{cases} \dfrac{\partial^2 y(t,x)}{\partial t^2} = 2\beta\Delta \dfrac{\partial y(t,x)}{\partial t} - \Delta^2 y(t,x) + u(t,x) + ay(t,x) + cu(t,x), \\ y(t,x) = \Delta y(t,x) = 0, \quad \text{on } (0,\tau) \times \partial\Omega, \\ y(0,x) = y_0(x), \quad y_t(x) = v_0(x), \quad x \in \Omega, \end{cases} \tag{1.3}$$

is approximately controllable.

After that, we write system (1.1) in the following form:

$$\begin{cases} \dfrac{\partial^2 y(t,x)}{\partial t^2} = 2\beta\Delta \dfrac{\partial y(t,x)}{\partial t} - \Delta^2 y(t,x) + u(t,x) \\ \qquad + ay(t,x) + cu(t,x) + g(t, y, y_t, u), \quad \text{in } (0,\tau) \times \Omega \\ y(t,x) = \Delta y(t,x) = 0, \quad \text{on } (0,\tau) \times \partial\Omega, \\ y(0,x) = y_0(x), \quad y_t(x) = v_0(x), \quad x \in \Omega, \end{cases} \tag{1.4}$$

where $g(t, y, v, u) = f(t, y, v, u) - ay - cu$ is a smooth-enough and bounded function.

The technique we use here to prove the controllability of the linear equation (1.3) is based on the Kalman condition [7] and Lemma 3.1.b from [6].

To prove the approximate controllability of the nonlinear system (1.1), we take some ideas from [14]. That is to say, the approximate controllability of system (1.1) follows from the approximate controllability of system (1.3), the compactness of the semigroup generated by linear part of system (1.3), the uniform boundedness of the nonlinear term g, and applying the Schauder fixed point theorem.

2 Abstract Formulation of the Problem

In this section, we choose the space in which this problem will be set as an abstract ordinary differential equation. Let $Z = L^2(\Omega)$ (with the norm $\|z\| = \|z\|_{L^2} =$

$\|z\|_Z, z \in Z)$ and consider the linear unbounded operator $A : D(A) \subset Z \to Z$ defined by $A\phi = -\Delta\phi$, where

$$D(A) = H_0^1(\Omega) \cap H^2(\Omega). \tag{2.1}$$

The operator A has the following very well-known properties: the spectrum of A consists of only eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty,$$

each one with multiplicity γ_n equal to the dimension of the corresponding eigenspace.

(a) There exists a complete orthonormal set $\{\phi_{n,k}\}_{k=1}^{\gamma_n}, n = 1, 2, 3, \ldots$ of eigenvectors of A.

(b) For all $z \in D(A)$, we have

$$Az = \sum_{n=1}^{\infty} \lambda_n \sum_{k=1}^{\gamma_n} < z, \phi_{n,k} > \phi_{n,k} = \sum_{n=1}^{\infty} \lambda_n E_n z, \tag{2.2}$$

where $< \cdot, \cdot >$ is the inner product in Z and

$$E_n z = \sum_{k=1}^{\gamma_n} < z, \phi_{n,k} > \phi_{n,k}. \tag{2.3}$$

So, $\{E_n\}$ is a family of complete orthogonal projections in z and

$$z = \sum_{n=1}^{\infty} E_n z, \quad z \in Z. \tag{2.4}$$

(c) $-A$ generates an analytic semigroup $\{T(t)\}_{t\geq 0}$ given by

$$T(t)z = \sum_{n=1}^{\infty} e^{-\lambda_n t} E_n z. \tag{2.5}$$

(d) The fractional powered spaces Z^r are given by

$$Z^r = D(A^r) = \left\{ z \in Z : \sum_{j=1}^{\infty} \lambda_j^{2r} \|E_j z\|^2 < \infty \right\}, \quad r \geq 0,$$

with the norm

$$\|z\|_r = \|A^r z\| = \left\{ \sum_{j=1}^{\infty} \lambda_j^{2r} \|E_j z\|^2 \right\}^{1/2}, \quad z \in Z^r,$$

and

$$A^r z = \sum_{j=1}^{\infty} \lambda_j^r E_j z. \tag{2.6}$$

For $r=1$, we define $Z_1 = Z^1 \times Z$, which is a Hilbert space with a norm given by

$$\left\| \begin{pmatrix} y \\ v \end{pmatrix} \right\|_{Z_1} = \left\{ \|y\|_1^2 + \|v\|^2 \right\}^{1/2}.$$

Hence, Eq. (1.1) can be written as an abstract system of ordinary differential equations in the Hilbert space Z_1 as follows:

$$\begin{cases} y' = v, \\ v' = -A^2 y - 2\beta A v + u + f(t, y, v, u). \end{cases} \tag{2.7}$$

Finally, Eq. (1.1) can be rewritten as a first-order system of ordinary differential equations in the Hilbert space Z_1 as follows:

$$z' = \mathcal{A}z + Bu + F(t, z, u), \;\; z \in Z_1, \;\; t \geq 0$$

where $u \in L^2([0, \tau]; U), \;\; U = Z = L^2(\Omega)$,

$$\mathcal{A} = \begin{pmatrix} 0 & I_Z \\ -A^2 & -2\beta A \end{pmatrix} \tag{2.8}$$

is an unbounded linear operator with domain

$$D(\mathcal{A}) = \{y \in H^4(\Omega) : \;\; y = \Delta y = 0 \;\; \text{on } (0, \tau) \times \partial\Omega\} \times D(A),$$

i.e.,

$$z = \begin{pmatrix} y \\ v \end{pmatrix} \in D(\mathcal{A}) \Leftrightarrow y \in \left\{ y \in H^4(\Omega) : \;\; y = \Delta y = 0 \;\; \text{on } (0, \tau) \times \partial\Omega \right\} \quad \text{and} \quad v \in D(A).$$

$I = I_Z : Z = L^2(\Omega) \to Z$ is the identity operator, $B : U \to Z_1$, $B = \begin{pmatrix} 0 \\ I_Z \end{pmatrix}$ is a bounded linear operator, and $F(t, z, u) = \begin{pmatrix} 0 \\ f(t, y, v, u) \end{pmatrix}$ is smooth enough with

$$\sup_{(t,z,u) \in \hat{Q}_\tau} \| F(t, z, u) - B^a z - B^c u \|_{Z_1} < \infty, \tag{2.9}$$

where $\hat{Q}_\tau = [0, \tau] \times Z_1 \times U$ and $B^a = \begin{pmatrix} 0 \\ aI \end{pmatrix}$ and $B^c = \begin{pmatrix} 0 \\ cI \end{pmatrix}$, $a < \lambda_1^2$.
Systems (1.3) and (1.4) can be written as an ordinary differential equation in the Hilbert space Z_1 as follows:

$$z' = \mathcal{A}z + Bu + B^a z + B^c u, \;\; t \in (0, \tau], \tag{2.10}$$

$$z' = \mathcal{A}z + Bu + B^a z + B^c u + G(t, z, u), \;\; t \in (0, \tau], \tag{2.11}$$

where $G(t, z, u) = F(t, z, u) - B^a z - B^c u$.
On the other hand, hypothesis (2.9) implies that

$$\sup_{(t,z,u) \in \hat{Q}_\tau} \| G(t, z, u) \|_{Z_1} < \infty. \tag{2.12}$$

Therefore, $G : \hat{Q}_\tau \longrightarrow Z_1$ is a smooth-enough and bounded function.
Now, using the following lemma from [10], we can prove that the linear unbounded operator $\mathcal{A}$ given by the linear equation (2.8) generates a strongly continuous compact semigroup in the space Z_1, which decays exponentially to zero.

Lemma 2.1 *Let Z be a Hilbert separable space and $\{A_j\}_{j\geq 1}$, $\{P_j\}_{j\geq 1}$ the two families of bounded linear operator in Z, with $\{P_j\}_{j\geq 1}$ a family of complete orthogonal projection such that*

$$A_j P_j = P_j A_j,\ j \geq 1.$$

Define the following family of linear operators:

$$T(t)z = \sum_{j=1}^{\infty} e^{A_j t} P_j z, \quad z \in Z, \quad t \geq 0.$$

Then:

(a) *$T(t)$ is a linear and bounded operator if $\|e^{A_j t}\| \leq g(t)$, $j = 1, 2, ...$, with $g(t) \geq 0$, continuous for $t \geq 0$.*
(b) *Under the same condition with that in (a), $\{T(t)\}_{t\geq 0}$ is a strongly continuous semigroup in the Hilbert space Z, whose infinitesimal generator A is given by*

$$Az = \sum_{j=1}^{\infty} A_j P_j z, \quad z \in D(A),$$

with

$$D(A) = \left\{ z \in Z : \sum_{j=1}^{\infty} \|A_j P_j z\|^2 < \infty \right\}.$$

(c) *The spectrum $\sigma(A)$ of A is given by*

$$\sigma(A) = \overline{\bigcup_{j=1}^{\infty} \sigma(\overline{A_j})},$$

where $\bar{A}_j = A_j P_j : \mathcal{R}(P_j) \to \mathcal{R}(P_j)$.

Theorem 2.1 *The operator $\mathcal{A}$ is the infinitesimal generator of a strongly continuous compact semigroup $\{T(t)\}_{t\geq 0}$ represented by*

$$T(t)z = \sum_{j=1}^{\infty} e^{\mathbb{A}_j t} P_j z,\ z \in Z_1,\ t \geq 0, \tag{2.13}$$

where $\{P_j\}_{j\geq 0}$ is a complete family of orthogonal projections in the Hilbert space Z_1 given by

$$P_j = diag(E_j, E_j), \tag{2.14}$$

and

$$\mathbb{A}_j = K_j P_j,\ K_j = \begin{pmatrix} 0 & 1 \\ -\lambda_j^2 & -2\beta\lambda_j \end{pmatrix} \quad j \geq 1,$$

and the adjoint operator $\mathbb{A}^*$ of the operator $\mathbb{A}$ is given by

$$\mathbb{A}_j^* = \tilde{K}_j P_j,\ \tilde{K}_j = \begin{pmatrix} 0 & -1 \\ \lambda_j^2 & -2\beta\lambda_j \end{pmatrix} \quad j \geq 1.$$

Moreover, the eigenvalues $\sigma_1(j),\ \sigma_2(j)$*, of the matrix* K_j *are simple and given by*

$$\sigma_1(j) = -\lambda_j\rho_1,\ \sigma_2(j) = -\lambda_j\rho_2$$

where $0 < \rho_1 < \rho_2$ *is given by*

$$\rho_1 = \beta - \sqrt{\beta^2 - 1} \quad \textit{and} \quad \rho_2 = \beta + \sqrt{\beta^2 - 1};\ ,\ \beta^2 > 1,$$

and this semigroup decays exponentially to zero

$$\| T(t) \| \leq Me^{-\mu t},\ t \geq 0,$$

where $\mu = \lambda_1\rho_1$ *and* $\| T(t) \| = sup_{\|z\|=1}\|T(t)z\|$.

Proof Let us compute $\mathcal{A}z$:

$$\begin{aligned}
\mathcal{A}z &= \begin{pmatrix} 0 & I \\ -A^2 & -2\beta A \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix} \\
&= \begin{pmatrix} v \\ -A^2y - 2\beta Av \end{pmatrix} \\
&= \begin{pmatrix} \sum_{j=1}^{\infty} E_jv \\ -\sum_{j=1}^{\infty}\lambda_j^2E_jy - 2\beta\sum_{j=1}^{\infty}\lambda_jE_jv \end{pmatrix} \\
&= \sum_{j=1}^{\infty}\begin{pmatrix} E_jv \\ -\lambda_j^2E_jy - 2\beta\lambda_jE_jv \end{pmatrix} \\
&= \sum_{j=1}^{\infty}\begin{pmatrix} 0 & 1 \\ -\lambda_j^2 & -2\beta\lambda_j \end{pmatrix}\begin{pmatrix} E_j & 0 \\ 0 & E_j \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix} \\
&= \sum_{j=1}^{\infty} A_jP_jz.
\end{aligned}$$

It is clear that $A_jP_j = P_jA_j$. Now, we need to check condition (a) from Lemma 2.1. To this end, we observe that the eigenvalues $\sigma_1(j),\ \sigma_2(j)$ of the matrix K_j are simple and given by

$$\sigma_1(j) = -\lambda_j\rho_1,\ \sigma_2(j) = -\lambda_j\rho_2$$

where $0 < \rho_1 < \rho_2$ are given by

$$\rho_1 = \beta - \sqrt{\beta^2 - 1} \quad \text{and} \quad \rho_2 = \beta + \sqrt{\beta^2 - 1};\quad \beta^2 > 1.$$

Since the eigenvalues of K_j are simple, there exists a complete family of complementary projections $\{q_i(j)\}_{i=1}^2$ in $I\!R^2$ such that

$$\begin{cases} K_j &= \sigma_1(j)q_1(j) + \sigma_1(j)q_2(j) \\ e^{K_jt} &= e^{-\lambda_j\rho_1t}q_1(j) + e^{-\lambda_j\rho_2t}q_2(j), \end{cases}$$

where $q_i(j), \quad i = 1, 2$, is given by

$$q_1(j) = \frac{1}{(\rho_1 - \rho_2)} \begin{pmatrix} \rho_2 & \frac{1}{\lambda_j} \\ -\lambda_j & \rho_2 - 2\beta \end{pmatrix}$$

$$q_2(j) = \frac{1}{(\rho_2 - \rho_1)} \begin{pmatrix} \rho_1 & \frac{1}{\lambda_j} \\ -\lambda_j & \rho_1 - 2\beta \end{pmatrix}.$$

Therefore,

$$\begin{cases} A_j &= \sigma_1(j) P_{j1} + \sigma_1(j) P_{j2} \\ e^{A_j t} &= e^{-\lambda_j \rho_1 t} P_{j1} + e^{-\lambda_j \rho_2 t} P_{j2}, \end{cases}$$

and

$$\mathcal{A}z = \sum_{j=1}^{\infty} \left\{ \sigma_1(j) P_{j1} z + \sigma_2(j) P_{j2} z \right\}, \tag{2.15}$$

where $P_{ji} = q_i(j) P_j$ is a complete family of orthogonal projections in Z_1.

To prove that $e^{A_n t} P_n : Z_1 \to Z_1$ satisfies condition (a) from Lemma 2.1, it will be enough to prove for example that $e^{-\lambda_n \rho_2 t} q_1(n) P_n, n = 1, 2, 3, \ldots$ satisfies condition (a). In fact, consider $z = (z_1, z_2,)^T \in Z_1$ such that $\|z\| = 1$. Then,

$$\|z_1\|_1^2 = \sum_{j=1}^{\infty} \lambda_j^2 \|E_j z_1\|^2 \leq 1, \quad \|z_2\|_Z^2 = \sum_{j=1}^{\infty} \|E_j z_2\|^2 \leq 1 \ .$$

Therefore, $\lambda_j \|E_j z_1\| \leq 1$, $\|E_j z_2\| \leq 1, \quad j = 1, 2, \ldots$. Then,

$$\begin{aligned} |e^{-\lambda_n \rho_1 t} q_1(n) P_n z\|_{Z_1}^2 &= \frac{e^{-2\lambda_n \rho_1 t}}{(\rho_1 - \rho_2)^2} \left\| \begin{matrix} \rho_2 E_n z_1 + \frac{1}{\lambda_n} E_n z_2 \\ -\lambda_n E_n z_1 + (\rho_2 - 2\beta) E_n z_2 \end{matrix} \right\|_{Z_1}^2 \\ &= \frac{e^{-2\lambda_n \rho_1 t}}{(\rho_1 - \rho_2)^2} \sum_{j=1}^{\infty} \lambda_j^2 \| E_j \left(\rho_2 E_n z_1 + \frac{1}{\lambda_n} E_n z_2 \right) \|^2 \\ &\quad + \frac{e^{-2\lambda_n \rho_1 t}}{(\rho_1 - \rho_2)^2} \sum_{j=1}^{\infty} \| E_j (-\lambda_n E_n z_1 + (\rho_2 - 2\beta) E_n z_2) \|^2 \\ &= \frac{e^{-2\lambda_n \rho_1 t}}{(\rho_1 - \rho_2)^2} \|\lambda_n \rho_2 E_n z_1 + \frac{1}{\lambda_n} E_n z_2\|^2 \\ &\quad + \frac{e^{-2\lambda_n \rho_1 t}}{(\rho_1 - \rho_2)^2} \| - \lambda_n E_n z_1 + (\rho_2 - 2\beta) E_n z_2\|^2. \end{aligned}$$

Since $\lambda_j \|E_j z_1\| \leq 1$ and $\|E_j z_2\| \leq 1, \quad j = 1, 2, \ldots$, we get that

$$|e^{-\lambda_n \rho_1 t} q_1(n) P_n z\|_{Z_1}^2 \leq M^2 e^{-2\lambda_n \rho_1 t}$$

where $M = M(\beta) \geq 1$ depending on β. Then, we have

$$\|e^{-\lambda_n \rho_1 t} q_1(n) P_n\|_{Z_1} \leq M(\beta) e^{-\lambda_n \rho_1 t}, \quad t \geq 0 \ \ n = 1, 2, \ldots.$$

In the same way, we obtain that

$$\|e^{-\lambda_n \rho_2 t} q_2(n) P_n\|_{Z_1} \leq M(\beta) e^{-\lambda_n \rho_2 t}, \;\; t \geq 0 \;\; n = 1, 2, \dots.$$

Therefore,

$$\|e^{A_n t} P_n\|_{Z_1} \leq M(\beta) e^{-\mu t}, \;\; t \geq 0 \;\; n = 1, 2, \dots,$$

was

$$\mu = \lambda_1 \rho_1.$$

Hence, applying Lemma 2.1, we obtain that $\mathcal{A}$ generates a strongly continuous semigroup given by Eq. (2.13), which implies that this semigroup is compact.
Next, we prove that this semigroup decays exponentially to zero. In fact,

$$\begin{aligned}
\|T(t)z\|^2 &= \sum_{j=1}^{\infty} \|e^{A_j t} P_j z\|^2 \\
&\leq \sum_{j=1}^{\infty} \|e^{A_j t}\|^2 \|P_j z\|^2 \\
&\leq M^2(\beta) e^{-2\mu t} \sum_{j=1}^{\infty} \|P_j z\|^2 \\
&= M^2(\beta) e^{-2\mu} \|z\|^2.
\end{aligned}$$

Therefore,

$$\|T(t)\| \leq M(\beta) e^{-\mu t}, \;\; t \geq 0.$$

□

Systems (2.10) and (2.11) also can be written as follows:

$$z' = \mathbb{A}z + (B + B^c)u, \;\; z \in Z_1 \; t \in (0, \tau]. \tag{2.16}$$

$$z' = \mathbb{A}z + (B + B^c)u + G(t, z, u), \;\; z \in Z_1 \; t \in (0, \tau] \tag{2.17}$$

where the operator $\mathbb{A}$ is given by

$$\mathbb{A} = \begin{pmatrix} 0 & I_Z \\ -A^2 + aI & -2\beta A \end{pmatrix}. \tag{2.18}$$

The following corollary follows from the foregoing theorem:

Corollary 2.1 *The operator $\mathbb{A}$ is the infinitesimal generator of a strongly continuous compact semigroup $\{T_a(t)\}_{t \geq 0}$ represented by*

$$T_a(t)z = \sum_{j=1}^{\infty} e^{\mathbb{A}_j t} P_j z, \;\; z \in Z_1, \;\; t \geq 0$$

where $\{P_j\}_{j\geq 0}$ *is a complete family of orthogonal projections in the Hilbert space* Z_1 *given by*

$$P_j = diag(E_j, E_j), \tag{2.19}$$

and

$$\mathbb{A}_j = K_j P_j, \;\; K_j = \begin{pmatrix} 0 & 1 \\ -\lambda_j^2 + a & -2\beta\lambda_j \end{pmatrix} \;\; j \geq 1,$$

and the adjoint operator $\mathbb{A}^*$ *of the operator* $\mathbb{A}$ *is given by*

$$\mathbb{A}_j^* = \tilde{K}_j P_j, \;\; \tilde{K}_j = \begin{pmatrix} 0 & \frac{a}{\lambda_j^2} - 1 \\ \lambda_j^2 & -2\beta\lambda_j \end{pmatrix} \;\; j \geq 1.$$

Moreover, the eigenvalues $\sigma_1(j),\ \sigma_2(j)$*, of the matrix* K_j *are simple and given by*

$$\sigma_1(j) = -\lambda_j\rho_{1j}, \;\; \sigma_2(j) = -\lambda_j\rho_{2j}$$

where $0 < \rho_{1j} < \rho_{2j}$ *is given by*

$$\rho_{1j} = \beta - \sqrt{\beta^2 - 1 + \frac{a}{\lambda_j^2}} \;\; and \;\; \rho_{2j} = \beta + \sqrt{\beta^2 - 1 + \frac{a}{\lambda_j^2}}; \;\; a < \lambda_1^2, \;\; \beta^2 > 1,$$

and this semigroup decays exponentially to zero

$$\| T_a(t) \| \leq Me^{-\mu t}, \;\; t \geq 0,$$

where $\mu = \lambda_1\rho_{11}$ *and* $\| T_a(t) \| = sup_{\|z\|=1} \| T_a(t)z \|$.

3 Approximate Controllability of the Linear System

In this section, we shall prove the approximate controllability of the linear system (2.16). To this end, for all $z_0 \in Z_1$ and $u \in L^2([0, \tau]; U)$, the initial value problem

$$\begin{cases} z'(t) = \mathbb{A}z(t) + (B + B^c)u(t), & z \in Z_1, \\ z(0) = z_0, \end{cases} \tag{3.1}$$

admits only one mild solution given by

$$z(t) = T_a(t)z_0 + \int_0^t T_a(t-s)(B + B^c)u(s)ds; \;\; t \in [0, \tau]. \tag{3.2}$$

Definition 3.1 For system (3.1), we define the following concept:
The controllability map (for $\tau > 0$) $G_a : L^2([0, \tau]; U) \longrightarrow Z_1$ is given by

$$G_a u = \int_0^\tau T_a(\tau - s)(B + B^c)u(s)ds, \tag{3.3}$$

whose adjoint operator $G_a^* : Z_1 \longrightarrow L^2([0, \tau]; U)$ is given by

$$(G_a^* z)(s) = (B^* + (B^c)^*)T_a(\tau - s)z, \;\; \forall s \in [0, \tau], \;\; \forall z \in Z_1. \tag{3.4}$$

The following lemma is trivial:

Lemma 3.1 *Equation 3.1 is approximately controllable on* $[0, \tau]$ *if, and only if,* $\overline{Rang(G_a)} = Z_1$.

The following theorem is a characterization of the approximate controllability of system (3.1) (see [1, 2, 15]):

Theorem 3.1 *System (3.1) is approximately controllable on* $[0, \tau]$ *if, and only if, any one of the following conditions hold:*

(a) $\overline{Rang(G_a)} = Z_1$.
(b) $Ker(G_a^*) = \{0\}$.
(c) $(B + B^c)^* T_a^*(s) z = 0, \ \forall s \in [0, \tau] \Longrightarrow z = 0$.
(d) $lim_{\alpha \to 0^+} \alpha(\alpha I + G_a G_a^*)^{-1} z = 0$.
(e) $sup_{\alpha > 0} \|\alpha(\alpha I + G_a G_a^*)^{-1}\| \leq 1$.
(f) $\langle G_a G_a^* z, z \rangle > 0, \ z \neq 0$ *in* Z.
(g) *For all* $z \in Z_1$, *we have* $G_a u_\alpha = z - \alpha(\alpha I + G_a G_a^*)^{-1} z$, *where*

$$u_\alpha = G_a^*(\alpha I + G_a G_a^*)^{-1} z, \ \alpha \in (0, 1].$$

So, $lim_{\alpha \longrightarrow 0} G_a u_\alpha = z$ *and the error* $E_\alpha z$ *of this approximation is given by*

$$E_\alpha z = \alpha(\alpha I + G_a G_a^*)^{-1} z, \ \alpha \in (0, 1].$$

Remark 3.1 Theorem 3.1 implies that the family of linear operators

$$\Gamma_\alpha z = (B + B^c)^\star T_a^*(\cdot)(\alpha I + G_a G_a^*)^{-1} z = G_a^*(\alpha I + G_a G_a^*)^{-1} z$$

is an approximate inverse for the right of the operator G_a in the sense that

$$\lim_{\alpha \longrightarrow 0} G_a \Gamma_\alpha z = z, \quad \forall z \in Z_1.$$

In other words,

$$\lim_{\alpha \longrightarrow 0} G_a \Gamma_\alpha = I,$$

in the strong topology of Z_1.

Now, we are ready to prove the controllability of the linear system (2.16).

Theorem 3.2 *System (3.1) is approximately controllable.*

Proof It is easy to see that $P_j(B + B^c)(B + B^c)^* = (B + B^c)(B + B^c)^* P_j, \ j = 1, 2, \ldots$. Then, applying Lemma 3.1. part (b) from [6], we obtain that the approximate controllability of system (3.1) is equivalent to the approximate controllability of each of the finite dimensional systems

$$y' = \mathbb{A}_j y + (B + B^c)_j u(t), \quad y(t) \in \text{Ran}(P_j), \ t \geq 0 \ j = 1, 2, \ldots, \tag{3.5}$$

where $(B+B^c)_j = P_j(B+B^c)_j : U \longrightarrow \mathrm{Ran}(P_j)$,

$$\begin{aligned}(B+B^c)_j u &= P_j(B+B^c)_j u\\ &= \begin{pmatrix} 0 \\ 1+c \end{pmatrix} E_j u\\ &= D_j E_j u.\end{aligned}$$

But, the approximate controllability of system (3.5) is equivalent to the controllability of each of finite dimensional systems

$$x' = C_j x + D_j u, \quad x \in \mathbb{R}^2, \tag{3.6}$$

where $u \in \mathbb{R}$ and

$$C_j = \begin{pmatrix} 0 & \lambda_j^2 \\ \frac{a}{\lambda_j^2} - 1 & -2\beta\lambda_j \end{pmatrix}, \qquad C_j^* = \begin{pmatrix} 0 & \frac{a}{\lambda_j^2} - 1 \\ \lambda_j^2 & -2\beta\lambda_j \end{pmatrix} = \tilde{K}_j, \quad j \geq 1.$$

And it is known (see [7, 9]) that system (3.6) is controllable if, and only if,

$$\mathrm{Rank}[D_j | C_j D_j] = 2,$$

which, doing the respective calculation, is trivially true.

In fact, the controllability of systems (3.5) and (3.6) are equivalent, respectively, to the following uniqueness continuation principle (see [1, 2]):

$$(B+B^c)_j^* e^{\mathbb{A}_j^* t} y = 0, \quad \forall t \in [0,\tau] \Rightarrow y = 0, \;\; y \in \mathrm{Ran}(P_j).$$

$$D_j^* e^{C_j^* t} x = 0, \quad \forall t \in [0,\tau] \Rightarrow x = 0, \;\; x \in \mathbb{R}^2.$$

On the other hand, we have that

$$(B+B^c)_j^* e^{\mathbb{A}_j^* t} y = D_j^* e^{\tilde{K}_j t} P_j y = D_j^* e^{C^* t} P_j y, \quad \forall t \in [0,\tau] \Rightarrow P_j y = y = 0.$$

In consequence, we have proved that system (3.1) is approximately controllable. □

4 Controllability of the Semilinear Beam Equation

In this section, we shall prove the main result of this paper, the controllability of the semilinear beam equation given (Eq. (1.1)), which is equivalent to prove the approximate controllability of system (2.10). To this end, we note that since the function $G(t,z,u)$ is smooth enough, then for all $z_0 \in Z_1$ and $u \in L^2([0,\tau];U)$ the initial value problem

$$\begin{cases} z'(t) = \mathbb{A}z + (B+B^c)u + G(t,z,u), \; t \geq 0, \\ z(0) = z_0, \end{cases} \tag{4.1}$$

admits only one *mild solution* given by (see [5], p. 90)

$$z(t) = T_a(t)z_0 + \int_0^t T_a(t-s)(B^c + B)u(s)ds + \int_0^t T_a(t-s)G(s, z, u)ds, \ \ t \in [0, \tau]. \tag{4.2}$$

Remark 4.1 (See [11, 12], and [13]) The function G is smooth enough if:

(a) The mild solution $z(u) = z_u$ of problem (4.1) is unique.
(b) The mild solution $z(u) = z_u$ depends continuously on u.
(c) Moreover, If F is a Lipschitz function, then $z(u) = z_u$, as a function of u, is also a Lipschitz function.

Remark 4.2 The approximate controllability of system (4.1) consists in what follows: Given two states $z_0, z_1 \in Z_1$, find a control $u \in L^2([0, \tau]; U)$ such that the corresponding mild solution (4.2) $z = z_u = z(u)$ satisfies

$$z_1 \approx T_a(\tau)z_0 + \int_0^\tau T_a(\tau-s)(B^c + B)u(s)ds + \int_0^\tau T_a(\tau-s)G(s, z_u, u)ds,$$

i.e.,

$$z_1 - T_a(\tau)z_0 \approx \int_0^\tau T_a(\tau-s)(B^c + B)u(s)ds + \int_0^\tau T_a(\tau-s)G(s, z_u, u)ds,$$

i.e.,

$$z_1 - T_a(\tau)z_0 \approx G_a(u) + \int_0^\tau T_a(\tau-s)G(s, z_u, u)ds,$$

i.e.,

$$G_a(u) \approx z_1 - T_a(\tau)z_0 - \int_0^\tau T_a(\tau-s)G(s, z_u, u)ds,$$

where the notation $l \approx m$ indicates that l is approximately equal to m.

Remark 4.3 Without lose of generality, from now on, we will assume that initial state z_0 is fixed.

These remarks serve as a motivation for the following definition:

Definition 4.1 For system (4.1), we define the following concept: The nonlinear controllability map (for $\tau > 0$) $G_g : L^2([0, \tau]; U) \longrightarrow Z_1$ is given by

$$G_g u = \int_0^\tau T_a(\tau-s)(B^c + B)u(s)ds + \int_0^\tau T_a(\tau-s)G(\tau, z_u, u)ds = G_a(u) + H(u)$$

where $H : L^2([0, \tau]; U) \longrightarrow Z_1$ is the nonlinear operator given by

$$H(u) = \int_0^\tau T_a(\tau-s)G(\tau, z_u, u)ds,$$

where $z = z_u = z(u)$ is the corresponding mild solution of the initial value problem (4.1).

The following lemma is trivial:

Lemma 4.1 *System* (4.1) *is approximately controllable on* $[0, \tau]$ *if, and only if,* $\overline{Rang(G_g)} = Z_1$.

Proof

(Sufficiency) Suppose system (4.1) is approximately controllable on $[0, \tau]$; i.e., for all $z_1 \in Z_1$ and $\epsilon > 0$, there exists a control $u \in L^2([0, \tau]; U)$ such that the corresponding mild solution of the initial value problem (4.1) satisfies

$$\|z(\tau) - z_1\|_{Z_1} \leq \epsilon.$$

Now, we shall prove that $\overline{\text{Rang}(G_g)} = Z_1$. In fact, if we put, for all $z \in Z_1$, $z_1 = z + T(\tau)z_0$, we obtain that

$$\|z(\tau) - z_1\|_{Z_1} = \|z(\tau) - T(\tau)z_0 - z\|_{Z_1} = \|G_g(u) - z\|_{Z_1} \leq \epsilon,$$

which implies that $\overline{\text{Rang}(G_g)} = Z_1$.

(Necessity) Suppose that $\overline{\text{Rang}(G_g)} = Z_1$; i.e., for all $\epsilon > 0$ and $z \in Z_1$, there exists a control $u \in L^2([0, \tau]; U)$ such that

$$\|G_g(u) - z\|_{Z_1} \leq \epsilon.$$

Now, if we put, for all $z_1 \in Z_1$ and $\epsilon > 0$, $z = z_1 - T_a(\tau)z_0$, we obtain that

$$\|G_g(u) - z\|_{Z_1} = \|G_g(u) + T(\tau)z_0 - z_1\|_{Z_1} = \|z(\tau) - z_1\|_{Z_1} \leq \epsilon,$$

which implies that system (4.1) is approximately controllable. □

From the above remarks and lemma, we have the following approximation:

$$G_a(u) \approx z_1 - T_a(\tau)z_0 - H(u) = z - H(u),$$

where $z = z_1 - T_a(\tau)z_0$. This motivates the following definition:

Definition 4.2 The following equation will be called the controllability equation associated to the nonlinear system (4.1)

$$u = \Gamma_\alpha(z - H(u)) = G_a^*(\alpha I + G_a G_a^*)^{-1}(z - H(u)), \ (0 < \alpha \leq 1).$$

Now, we are ready to present and prove the main result of this paper, which is the approximate controllability of the semilinear beam equation (1.1)

Theorem 4.1 *System* (4.1) *is approximately controllable on* $[0, \tau]$. *Moreover, a sequence of controls steering system* (4.1) *from initial state* z_0 *to an* ϵ*-neighborhood of the final state* z_1 *at time* $\tau > 0$ *is given by*

$$u_\alpha(t) = (B + B^c)^* T_a^*(\tau - t)(\alpha I + G_a G_a^*)^{-1}(z_1 - T_a(\tau)z_0 - H(u_\alpha)),$$

and the error of this approximation E_α *is given by*

$$E_\alpha = \alpha(\alpha I + G_a G_a^*)^{-1}(z_1 - T_a(\tau)z_0 - H(u_\alpha)).$$

Proof For each $z \in Z_1$ fixed, we shall consider the following family of nonlinear operators $K_\alpha : L^2([0,\tau];U) \longrightarrow L^2([0,\tau];U)$ given by

$$K_\alpha(u) = \Gamma_\alpha(z - H(u)) = G_a^*(\alpha I + G_a G_a^*)^{-1}(z - H(u)), \ \ (0 < \alpha \leq 1).$$

First, we shall prove that for all $\alpha \in (0,1]$, the operator K_α has a fixed point u_α. In fact, since the semigroup $\{T_a(t)\}_{t\geq 0}$ is compact, then using the result from [3], the smoothness and the boundedness of the nonlinear term G, we obtain that the operator H is compact and the set $\overline{\text{Rang}(H)}$ is compact.

On the other hand, since G is bounded and $\| T_a(t) \| \leq Me^{-\mu t}, \ t \geq 0$, there exists a constant $R > 0$ such that

$$\| H(u) \| \leq R, \quad \forall u \in L^2(0,\tau;U).$$

Then,

$$\| K_\alpha(u) \| \leq \| \Gamma_\alpha \| (\| z \| + R), \quad \forall u \in L^2(0,\tau;U).$$

Therefore, the operator K_α maps the ball $B_r(0) \subset L^2(0,\tau;U)$ of center zero and radio $r \geq \| \Gamma_\alpha \| (\| z \| + R)$ into itself. Hence, applying the Schauder fixed point theorem, we get that the operator K_α has a fixed point $u_\alpha \in B_r(0) \subset L^2([0,\tau];U)$.

Since $\overline{\text{Rang}(H)}$ is compact, without loss of generality, we can assume that the sequence $H(u_\alpha)$ converges to $y \in Z_1$. So, if

$$u_\alpha = \Gamma_\alpha(z - H(u_\alpha)) = G_a^*(\alpha I + G_a G_a^*)^{-1}(z - H(u_\alpha)),$$

then

$$\begin{aligned} G_a u_\alpha &= G_a \Gamma_\alpha(z - H(u_\alpha)) = G_a G_a^*(\alpha I + G_a G_a^*)^{-1}(z - H(u_\alpha)) \\ &= (\alpha I + G_a G_a^* - \alpha I)(\alpha I + G_a G_a^*)^{-1}(z - H(u_\alpha)) \\ &= z - H(u_\alpha) - \alpha(\alpha I + G_a G_a^*)^{-1}(z - H(u_\alpha)). \end{aligned}$$

Hence,

$$G_a u_\alpha + H(u_\alpha) = z - \alpha(\alpha I + G_a G_a^*)^{-1}(z - H(u_\alpha)).$$

To conclude the proof of this theorem, it is enough to prove that

$$\lim_{\alpha \longrightarrow 0} \left\{ -\alpha(\alpha I + G_a G_a^*)^{-1}(z - H(u_\alpha)) \right\} = 0.$$

From Theorem 3.1 part (d), we get that

$$\begin{aligned} \lim_{\alpha \longrightarrow 0} \left\{ -\alpha(\alpha I + G_a G_a^*)^{-1}(z - H(u_\alpha)) \right\} &= - \lim_{\alpha \longrightarrow 0} \left\{ -\alpha(\alpha I + G_a G_a^*)^{-1} H(u_\alpha) \right\} \\ &= - \lim_{\alpha \longrightarrow 0} \left\{ -\alpha(\alpha I + G_a G_a^*)^{-1}(H(u_\alpha) - y) \right\} \\ &\quad - \lim_{\alpha \longrightarrow 0} \left\{ -\alpha(\alpha I + G_a G_a^*)^{-1} y \right\} \\ &= - \lim_{\alpha \longrightarrow 0} \left\{ -\alpha(\alpha I + G_a G_a^*)^{-1}(H(u_\alpha) - y) \right\}. \end{aligned}$$

On the other hand, from Theorem 3.1 part (e), we get that

$$\| \alpha \left(\alpha I + G_a G_a^*\right)^{-1} (H(u_\alpha) - y) \| \leq \| H(u_\alpha) - y \| .$$

Therefore, since $H(u_\alpha)$ converges to y, we get that

$$\lim_{\alpha \longrightarrow 0} \left\{ -\alpha(\alpha I + G_a G_a^*)^{-1}(H(u_\alpha) - y) \right\} = 0.$$

Consequently,

$$\lim_{\alpha \longrightarrow 0} \left\{ -\alpha(\alpha I + G_a G_a^*)^{-1}(z - H(u_\alpha)) \right\} = 0.$$

So, putting $z = z_1 - T_a(\tau)z_0$ and using Eq. (4.2), we obtain the following nice expression:

$$z_1 = \lim_{\alpha \longrightarrow 0} \left\{ T_a(\tau)z_0 + \int_0^\tau T_a(\tau - s)(B^c + B)u_\alpha(s)ds + \int_0^\tau T_a(\tau - s)G(s, z_{u_\alpha}, u_\alpha)ds \right\}.$$

□

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